▸ DENIZ TAHMOURESI, MAJID ALIZADE, PHILIPPE BALBIANI, MOJTABA MOJTAHEDI, *Unification for weak classical Modal logics*.
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In classical propositional logic (CL), constructed with a complete set of definitional connectives, many ideal properties of propositional logic are realized. For example, its satisfiable formulas form an NP-complete set ([1], chapter 9), and all admissible rules in CL are derivable [2]: chapter 1. However, these properties can be altered under syntactic restrictions–as in [3] and [4]–or when axioms and inference rules are weakened or extended, as in Intuitionistic logic (IPC) and intermediate logics, or when new semantical operators are introduced as in modal logics. For instance, in modal logics such as $S4$, the satisfiable formulas are PSPACE-complete ([5], chapter 6), and some admissible rules become non-derivable ([6], chapter 3).

Unification is important because, in a decidable logic, the existence of a computable finite complete set of unifiers allows for an effective decision procedure for the admissibility of inference rules. Unifiability in logical systems is commonly classified as follows: a formula is *finitary* if it has a minimal finite complete set of unifiers, *unitary* if this set reduces to a singleton, *infinitary* if it is not unitary or finitary and for every unifiable formula, there is a complete set of maximal unifiers of it, and *nullary* if it is not either of above cases. Classical propositional logic (CL), when based on a complete set of definitional connectives, is unitary and has NP-complete unification complexity ([7], [9], [10], [8]). In contrast, the implication-negation fragment of intuitionistic propositional logic (IPC) is finitary [12], and the modal logic K is nullary [11]. These contrasts, motivate our investigation of how the unification properties of CL behave when extended with a modal operator $\Box$, but without imposing any axiom schema for it. We refer to this weak classical modal logic as $\mathsf{CL}_\Box$, and our goal is to analyze its unification type.

Since the operator $\Box$ is introduced without semantic constraints, one might treat each boxed formula in $\mathsf{CL}_\Box$ as a parameter, in the sense of parametric classical propositional logic ($\mathsf{CL}_{\mathsf{par}}$). This suggests that the unification behavior of $\mathsf{CL}_\Box$ could closely resemble that of $\mathsf{CL}_{\mathsf{par}}$, and one might attempt to establish a one-to-one correspondence between unifiers of a formula A in $\mathsf{CL}_\Box$ and unifiers of its corresponding formula $\bar{A}$ in $\mathsf{CL}_{\mathsf{par}}$. However, this correspondence fails in general. Consider the formula $A = x \to \Box x$ in $\mathsf{CL}_\Box$, which maps to $\bar{A} = x \to p$ in $\mathsf{CL}_{\mathsf{par}}$, with x a variable and p a parameter. Define a substitution θ_0 by $\theta_0(x) := x \wedge p$. This θ_0 is a valid unifier of $\bar{A}$ in $\mathsf{CL}_{\mathsf{par}}$. However, the analogous substitution in $\mathsf{CL}_\Box$, defined by $\theta(x) := x \wedge \Box x$, fails to unify A. This example illustrates that $\mathsf{CL}_\Box$ and $\mathsf{CL}_{\mathsf{par}}$ differ in essential ways, and a more nuanced analysis is required.

To determine the unification type of $\mathsf{CL}_\Box$, it is essential to observe that formulas of the form $\Box A \to \Box B$, and similar structures, are unifiable only when there exists a substitution θ such that $\theta(A) = \theta(B)$. These particular syntactic forms play a central role in the analysis. Consequently, addressing the unification problem in $\mathsf{CL}_\Box$ requires a syntactic approach rather than a purely semantic one.

Even after analyzing the syntactic unification type of the problem and identifying a complete set of syntactic unifiers, combining this approach with the known method for $\mathsf{CL_{par}}$ to compute the general unifiers does not lead to a satisfactory result. For example, consider the formula $A = (x \to \Box y) \wedge (y \to \Box x)$. Combining the syntactic unifier with the Ghilardi approach yields a general substitution for $A = (x \to \Box y) \wedge (y \to \Box x)$ as $\theta(x) := A \wedge x$ and $\theta(y) := A \wedge y$. Yet, the substitution is not a unifier for A. Specifically, since $\Box x$ appears in $\theta(y)$ and $\Box y$ appears in $\theta(x)$, the resulting substitution is circular (not well-founded in this context). This indicates that the unification problem in $\mathsf{CL}_\Box$ must be addressed by introducing a restricted form of unification that avoids such circular dependencies.

If $\mathsf{CL}_\Box^+$ is defined as $\mathsf{CL}_\Box$ extended with the following inference rule (Weak Necessitation Rule):

$$\mathsf{wN}\ \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

then, by setting aside the notion of syntactic unification–since it is not effective in this context–and employing well-founded unifiers, we show that the unification type of $\mathsf{CL}_\Box^+$ is finitary.

However, such a claim does not hold for $\mathsf{CL}_\Box$. For example, let $A = (x \to \Box y) \wedge (y \to \Box x)$. As before, by interpreting the boxed formulas as parameters, consider a substitution θ defined by $\theta(x) := \bot$, $\theta(y) := \Box\bot$, and another substitution λ where $\lambda(x) := \bot \wedge \bot$ and $\lambda(y) := \Box\bot$. Although $\mathsf{CL}_\Box \vdash \theta(x) \leftrightarrow \lambda(x)$ and $\mathsf{CL}_\Box \vdash \theta(y) \leftrightarrow \lambda(y)$, θ unifies A while λ does not. This demonstrates that, in the absence of the Weak Necessitation Rule–and even when syntactic unification is considered–the unification problem in $\mathsf{CL}_\Box$ becomes significantly more intricate and the result is an infinite set of unifiers. This complexity motivates our investigation and is the central focus of the present work, which explores these issues in depth.

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