▸ ANTONIO PICCOLOMINI D'ARAGONA, *On the non-classical, non-monotonic and uniform proof-theoretic validity of weak excluded middle*.
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Proof-theoretic semantics (PTS) is dealt with in essentially two ways [14, 15]. One recent approach, called *base*(*-extension*) *semantics* (B-eS), focuses on a proof-based notion of *consequence* between sets of formulas Γ and formulas A. In Prawitz's original picture from the early 70s [9, 10, 11], now called *proof-theoretic validity* (P-tV), consequence is instead taken to be a *derived* notion only. Here, the prior concept is that of *valid proof-structure*. A is said to be consequence of Γ if there is a valid proof-structure from assumptions Γ to conclusion A.

A proof-structure is a formula-tree in the Natural Deduction style, where *arbitrary* inferences (with potential dischargements) are applied. The validity of the proof-structure is equated with its *reducibility* to a privileged form, modulo a set $\mathfrak{J}$ of *proof-rewriting functions*. What the privileged form is depends on the rules one takes to be valid by default. These normally are the *introductions*, so the proof-structure must reduce to one which ends by introduction, whose sub-structures are also valid modulo $\mathfrak{J}$. The same holds for B-eS, where the clauses defining the consequence relation will typically mirror the Natural Deduction introductions, i.e., they will state the *conditions* for formulas of different kinds to be valid. Elimination-based approaches are also available, but I shall not deal with them in my talk—see, e.g., [1, 2, 12, 18].

Both B-eS and P-tV start with local notions of consequence or validity *over an atomic proof-system*, i.e., over a set of higher-level atomic rules [13]. *Logical* consequence and *logical* validity obtain from the local notions by universally quantifying over all atomic proof-systems (of a given class).

The local notions can be defined either monotonically or non-monotonically. Monotonicity requires that, if A is consequence of Γ over an atomic proof-system $\mathfrak{B}$, or if the proof-structure π is valid over $\mathfrak{B}$, then A is consequence of Γ, or π is valid, over all atomic proof-systems (of a given class) $\mathfrak{C} \supseteq \mathfrak{B}$. This instead fails in the non-monotonic approach.

The link between the B-eS and the P-tV notions of consequence is not trivial. P-tV seems to be more intensional, since here the holding of consequence, which takes place at a purely sentential level in B-eS, must be witnessed by suitable proof-structures and proof-rewriting functions. One could hence wonder whether the two notions coincide, in particular whether (in)completeness results from one approach can be transferred to the other.

Many completeness and incompleteness results have been obtained for monotonic B-eS and P-tV. Intuitionistic logic (`IL`) is known to be incomplete over both of them [6, 7]—in the case of B-eS, even when the consequence relation is closed under replacements of atoms with formulas [8]. As a positive result, B-eS with atomic proof-systems of level ≥ 2 can be proved to be a semantics for inquisitive logic [16], and for `IL` too when the theory of atomic proof-systems is suitably modified [15, 17]. Monotonic P-tV can be shown to be equivalent to monotonic B-eS under the following conditions: (1) the consequence relation is restricted to finite Γ-s, and (2) the sets of rewriting functions which determine the reducibility of proof-structures are non-uniform, i.e., the output values that their functions produce cannot be defined schematically, and independently of considerations about the validity of the input values on given atomic proof-systems (similarly to standard proof-rewriting functions used in proofs of normalisation in Natural Deduction) [4, 16]. So, `IL` is incomplete over this specific form of P-tV—which we may call *liberal proof-theoretic validity* (LP-tV)—also when consequence is closed under replacements. This is because the incompleteness proof of `IL` over B-eS goes through already when one restricts oneself to finite Γ-s [4].

As for non-monotonic B-eS and P-tV, it has been proved that classical logic is sound and complete over the former with classical logic in the meta-language [15], and that it is sound relative to non-monotonic LP-tV under the same condition [3].

Concerning the last result, the required liberality of the proof-rewriting functions amounts to what follows. One proves classically that, for every atomic proof-system $\mathfrak{P}$ in a class of atomic proof-systems $\mathbb{P}$, Excluded Middle (EM) $A \vee \neg A$ can be rewritten either via the empty function $\phi_\emptyset$, so EM becomes

$$\dfrac{\dfrac{\dfrac{[A]}{\bot}}{\neg A}}{A \vee \neg A}$$

or via a function $\phi_\mathfrak{P}$ which picks a categorical proof π for A on $\mathfrak{P}$, so EM becomes

$$\dfrac{\begin{array}{c}\pi \\ A\end{array}}{A \vee \neg A}$$

Thus, EM is logically valid, namely, valid on every $\mathfrak{P} \in \mathbb{P}$, relative to the set of proof-rewriting functions $\Pi = \{\phi_\emptyset\} \cup \bigcup_{\mathfrak{P} \in \mathbb{P}} \{\phi_\mathfrak{P}\}$. However, Π is non-uniform in the sense hinted at above: the output values of its elements will depend each time on specific atomic proof-systems, so $\mathfrak{P}_1 \neq \mathfrak{P}_2$ might imply $\phi_{\mathfrak{P}_1} \neq \phi_{\mathfrak{P}_2}$, and we may not be able to define schematically what these values are.

Therefore, the result is only partly satisfactory, not just because it requires classical meta-logic, but also because, in Prawitz's approach, sets of proof-rewriting functions are meant to be uniform [3, 10, 15]. Non-monotonic B-eS can be shown to be equivalent to non-monotonic LP-tV *without classical meta-logic* [5], but this leaves the non-uniformity issue unsolved.

Schroeder-Heister has remarked that no special problem stems from the fact that the incompleteness of IL over non-monotonic B-eS obtains classically. An intuitionistic proof cannot be available, for this would imply "a classical contradiction. Given that these proofs can be coded in first-order arithmetic and that classical arithmetic and Heyting arithmetic are equiconsistent, such a claim cannot be upheld" [15, p. 501]. In the case of LP-tV, though, classical logic alone is not enough for proving the incompleteness of IL, since one also needs the requirement of non-uniformity on sets of proof-rewriting functions. So, one may be led to think that Schroeder-Heister's observation does not apply to P-tV—i.e., if one insists that sets of proof-rewriting functions *should be* uniform. But this is not the case, and in fact, one does not even need classical meta-logic, since the following holds.

Fact. Weak Excluded Middle (WEM) $\neg A \vee \neg\neg A$ is logically valid relative to a *uniform* set of proof-rewriting functions, if WEM itself holds in the meta-logic.

Thus, in order to make P-tV more constructive, one should put more constraints on sets of rewriting functions. One option might be to ask, e.g., that these sets give rise to a *convergent* reducibility relation, which generalises the Church-Rosser property in typed λ-calculi or—via the Curry-Howard isomorphism—in Natural Deduction systems. I shall conclude with discussions on the way in which such a variant of non-monotonic P-tV could be developed, and on the behaviour it may have with respect to (in)completeness issues.

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