▸ ANDREA VOLPI, *The barrier Ramsey theorem.*
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It is known since Gödel's first incompleteness theorem that there are true statements about the standard natural numbers that cannot be proved with the axioms of Peano arithmetic. In [11] Paris and Harrington proved that a certain statement in finite Ramsey theory, expressible in Peano arithmetic, is not provable in this system. This has been claimed to be the first "natural" statement independent from Peano arithmetic.

THEOREM 1 (Paris-Harrington 1977). *Let $n, k, m \in \mathbb{N}$ be such that $m \geq n$. There exists $N \in \mathbb{N}$ such that for each $s \subseteq \mathbb{N}$ of cardinality N and each coloring of the n-size subsets of s in k colors, there exists a set $t \subseteq s$ which is homogeneous for the coloring and such that $|t| > \max(m, \min t)$.*

The stress is on the fact that the largeness of the homogeneous set is measured in terms of its elements, and not only of its cardinality. After this seminal result, Ketonen and Solovay [7] introduced the notion of α-largeness (for $\alpha < \epsilon_0$) which generalizes this idea.

Given a countable ordinal Γ for which we have a system of fundamental sequences, we can define what it means for a finite set of natural numbers to be α-large for $\alpha < \Gamma$. Here we mention that typically (though not always, since largeness depends on the chosen system) a set s is n-large (for $n \in \omega$) when $|s| \geq n$ and ω-large if $|s| \geq \min s + 1$. In this parlance, Theorem 1 states the existence of homogeneous ω-large sets for k-colorings of n-large subsets of a N-large set.

More recently, various authors considered statements extending Theorem 1 to largeness notions and established some relationships between the various parameters (see e.g. [1, 2, 3, 9, 10, 8]). All of these results consider α-largeness notions for $\alpha < \epsilon_0$ and finite Ramsey like statements that restrict the sizes of the tuples being colored to be n for some fixed natural number n. We aim to extend the previous results by studying colorings of all γ-size subsets of some finite set s also when $\gamma \geq \omega$. In [12, 5, 4] the infinite Ramsey theorem has been extended to colorings of the γ-size subsets of an infinite set (the homogeneous set is required to be infinite).

We work with a more flexible framework: largeness notions induced by blocks and barriers. These generalize largeness notions induced by systems of fundamental sequences. Moreover, each block has a countable ordinal associated to it, its height, which measures the complexity of the block and of its largeness notion.

Notice also that a system of fundamental sequences is defined on some (typically constructive) ordinal Γ and once we fix the system we can work only with ordinals below Γ. On the other hand, we can consider blocks with arbitrary (countable) height.

A useful tool to state results in Ramsey theory is the arrow notation: we adapt this notation to our framework and write $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$ where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are blocks and $k \in \mathbb{N}$. Then, given ordinals α and γ, we define $\mathrm{Ram}(\alpha)^{\gamma}_k$ as the least β such that for each $\mathcal{A}$ and $\mathcal{C}$ of height respectively α and γ there exists $\mathcal{B}$ of height β with $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$.

Our main result is that, for $\gamma < \alpha$ with α infinite, the ordinal $\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega} = \sup_{k\in\mathbb{N}} \mathrm{Ram}(\alpha)^{1+\gamma}_k$ is

$$\varphi_{\log \gamma}(\alpha \cdot \omega),$$

where $\varphi_{\log \gamma}$ denotes a composition of Veblen functions indexed by the ordinals resulting from a logarithm of γ obtained from its Cantor normal form (it is called a hyperation of ordinal exponentiation in base ω in [6, Corollary 4.10]).

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