▸ SOFÍA SANTIAGO-FERNÁNDEZ, DAVID FERNÁNDEZ-DUQUE, AND JOOST J. JOOSTEN, *Rewrite normalization in tree rewriting calculi for strictly positive logics.*
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§1. Introduction. Strictly positive logics (spi-logics) are fragments of modal logic defined over the language $\mathscr{L}^+$, which allows formulas constructed from propositional variables, the constant $\top$, conjunction, and diamond modalities. Inspired by the tree-based representation of strictly positive formulas introduced by Kikot et al. [2] and Beklemishev [3], recent work [8] shows that certain spi-logics can be effectively captured using tree rewriting systems. This approach offers new tools for analyzing provability, based on a correspondence between $\mathscr{L}^+$ formulas and inductively defined *modal trees*.

We extend the framework developed in [8] to a broader family of spi-logics with basic system $\mathbf{K}^+$ [7], the strictly positive fragment of polymodal $\mathbf{K}$. Our approach includes extensions of $\mathbf{K}^+$ obtained by adding standard axioms such as monotonicity, transitivity, and the hierarchy-sensitive interaction axiom (J):

$$(\mathsf{J}) \quad \langle\alpha\rangle\varphi \wedge \langle\beta\rangle\psi \vdash \langle\alpha\rangle(\varphi \wedge \langle\beta\rangle\psi), \quad \text{for } \alpha > \beta. \tag{1}$$

which governs the interplay between modalities of different strengths. One important such extension is $\mathbf{K4}^+$ [2], the strictly positive fragment of $\mathbf{K4}$; while the most powerful one is the *Reflection Calculus* ($\mathbf{RC}$, cf. [6], [5]), which corresponds to the strictly positive fragment of polymodal $\mathbf{GLP}$ [4].

The rewriting calculi associated with this family of spi-logics are defined by rules that transform modal trees through subtree replacement. These rules fall into five kinds according to the transformation they perform: atomic, structural, replicative, decreasing and modal. Atomic rules modify node labels by adding or removing propositional variables; the structural rule rearranges the order of children; the replicative rule duplicates the child of a node; decreasing rules remove children and enforce transitivity; and modal rules decrease edge labels and restructure the tree to reflect the behavior of the axiom (J).

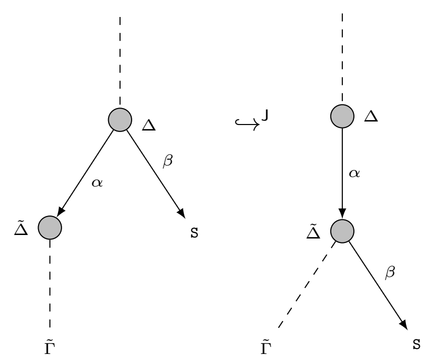

FIGURE 1. J-rule. The dash lines indicate portions of the tree that are not affected by the rule.

§2. Main results. Our principal contribution is the rewrite normalization of the tree rewriting calculi for the considered spi-logics, providing explicit upper bounds that reflect the interactions among the rewriting rules. This normalization reorganizes tree transformations into a canonical form, consisting of a structured sequence of rule applications grouped by their respective kinds. More precisely, given a rewriting sequence Ω, it can be transformed equivalently into a normal rewriting sequence of the form:

$$\Omega_+ \frown \Omega_\diamond \frown \Omega_\delta \frown \Omega_\rho \frown \Omega_\sigma, \tag{2}$$

where $\Omega_+, \Omega_\diamond, \Omega_\delta, \Omega_\rho$ and Ω_σ are, respectively, sequences of replicative, modal, decreasing, atomic and structural rules.

To achieve normalization, our approach follows two main directions: (i) analyzing how tree metrics change under rewriting to establish an appropriate bounding framework, and (ii) studying the permutation of rule applications. Permuting rules generally involves intricate technical reasoning about tree positions and the conditions governing rule application. When two rules are applied at disjoint positions, their applications commute straightforwardly. However, overlapping applications may require introducing additional rules to resolve conflicts. In particular, when the rule for subtree duplication is involved, permuting it with a sequence of other rule applications typically leads to an exponential increase in the total number of rules. For tree rewriting calculi of spi-logics without axiom (J), normalization exhibits exponential bounds:

$$|\Omega_+| \leq |\Omega|, \quad |\Omega_\diamond| \leq 2^{|\Omega_\delta|+|\Omega|}, \quad |\Omega_\delta \frown \Omega_\rho| \leq 2^{|\Omega|}, \quad |\Omega_\sigma| \leq \mathsf{n}(\mathtt{S}) - 1, \tag{3}$$

where $\mathsf{n}(\mathtt{S})$ denotes the number of nodes in the target tree.

In contrast, for logics including (J), normalization cannot be achieved by uniquely analyzing the combinatorial scenarios of rule applications, as was possible in the absence of J. Specifically, the J-rule introduces complex interactions with replicative rules that perform subtree duplication, causing the normalization process to exhibit double-exponential complexity. To address the subtle challenge, we introduce the notion of J-*flags*, which track subtrees affected by J throughout the rewriting process. These flags allow us to show that sequences of replicative rules permute with the application J-rules by induction on the number of J-flags appearing in the resulting tree. Notably, we demonstrate that this is a well-founded inductive argument since, after intermediate rearrangements on the application of the rules, the number of J-flags introduced is strictly smaller than in the original derivation.

When considering spi-logics including axiom (J), the upper bound for the replicative and modal rules of their corresponding tree rewriting calculi satisfy:

$$|\Omega_+| \leq (\mathsf{w}(\mathtt{T})+1)^{(\mathsf{h}(\mathtt{T})+1)^{2\cdot|\Omega|}}, \quad |\Omega_\diamond| \leq 2^{|\Omega_\delta|+2|\Omega|(\,|\Omega_+|+\mathsf{w}(\mathtt{T})+1)^{\mathsf{h}(\mathtt{T})}}, \tag{4}$$

where $\mathsf{w}(\mathtt{T})$ and $\mathsf{h}(\mathtt{T})$ denote the width and height of the initial tree $\mathtt{T}$.

In conclusion, we hope the presented results contribute to the theoretical foundations of spi-logics. The tree rewriting view not only captures the proof-theoretic essence of spi-logics but also offers an effective method for understanding modal interaction through normalization. Beyond their theoretical value, our findings support a more efficient approach to proof search by significantly reducing redundancy: thanks to the normalization theorems, it suffices to consider only normalized rewriting sequences. This restriction narrows the search space and improves computational efficiency [1].

[1] Goubault-Larrecq, Jean and Mackie, Ian, ***Proof theory and automated deduction***, Springer Science & Business Media, 2001.

[2] Kikot, Stanislav and Kurucz, Agi and Tanaka, Yoshihito and Wolter, Frank and Zakharyaschev, Michael, *Kripke completeness of strictly positive modal logics over meet-semilattices with operators*, ***The Journal of Symbolic Logic***, vol. 84 (2019), no. 2, pp. 533–588.

[3] Lev D. Beklemishev, *Positive provability logic for uniform reflection principles*, ***Annals of Pure and Applied Logic***, vol. 165 (2014), no. 1, pp. 82–105.

[4] Japaridze, G., *The polymodal provability logic*, ***Intensional logics and logical structure of theories: material from the Fourth Soviet-Finnish Symposium on Logic*** Metsniereba, Telavi, 1988.

[5] Dashkov, Evgenij Vladimirovich, *On the positive fragment of the polymodal provability logic GLP*, ***Mathematical Notes***, vol. 91 (2012), pp. 318–333.

[6] Lev D. Beklemishev, *Calibrating provability logic*, ***Advances in modal logic*** (T. Bolander, T. Braüner, T. S. Ghilardi and L. Moss), College Publications, London, 2012, pp. 89–94.

[7] Lev D. Beklemishev, *A note on strictly positive logics and word rewriting systems*, ***Larisa Maksimova on Implication, Interpolation, and Definability*** (Sergei Odintsov), Springer Verlag, 2018.

[8] Sofía Santiago-Fernández and Joost J. Joosten and David Fernández-Duque, *A tree rewriting system for the Reflection Calculus*, ***Advances in Modal Logic*** (Agata Ciabattoni, David Gabelaia and Ivan Sedlár), College Publications, London, 2024, pp. 675–695.