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Predicative recursion on the Veblen hierarchy.
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Bellantoni and Cook [1] characterized linear space (i.e., the class $\mathcal{E}_2$) and polynomial-time computable functions by a restricted form of recursion on numbers and on binary strings respectively. This approach to recursion is called *predicative recursion*, and is inspired by the stance in the foundations of mathematics known as *predicativism*, which rejects any definition that refers to itself. The main idea is to have functions with two sorts of inputs, one for recursion (*normal* inputs), and another for composition (*safe* inputs). In recursion, values of the function on smaller inputs are allowed only in safe positions, and in composition no safe inputs are allowed in normal position. Based on that, many other well-known complexity classes have been characterized in the literature. These characterizations are machine-independent, free of explicit bounds, and provide insights on the structural form of the computations. For example, Bellantoni [2] introduced a predicative form of minimization to capture all levels of the polynomial-time hierarchy; Leivant [4] employed predicative recursion on finite-type functionals to characterize the class $\mathcal{E}_3$ of elementary functions; and Wirz [5] extended predicative recursion by adding more sorts of inputs to classify the Grzegorczyk hierarchy $\{\mathcal{E}_k\}_{k\geq 2}$.

This widespread use of *predicativization* motivates predicative recursion in its most general form, i.e., predicative recursion on an arbitrary well-founded structure to harness the recursive computational power specific to that structure. While compelling individual results link predicativization and computation, a *systematic* exploration of this relationship is underdeveloped.

In this work, we generalize predicative recursion to constructive ordinals (as representatives of arbitrary well-founded structures) and characterize such recursion up to the ordinals in the Veblen hierarchy, using the Grzegorczyk hierarchy. In the following, we present the details of such characterization.

First, we recall that constructive ordinals [3], or simply *ordinals* for us, are rooted, well-founded trees with either a single or infinitely countable branching at each node. A single node is denoted by 0; the result of a single branching on α is denoted by $\alpha+1$ and called a *successor*; and the result of an $\mathbb{N}$-indexed branching is denoted by $\langle\mu_i\rangle_{i\in\mathbb{N}}$ and called a *limit*. The set of constructive ordinals is denoted by Ω. Constructive ordinals are ordered by the subtree relation $\prec$, which is well-founded. For any $\alpha \in \Omega$, we let $\mathsf{D}_\alpha := \{\beta \in \Omega \mid \beta \prec \alpha\}$. Moreover, any downward-closed set $\mathsf{A} \subseteq \Omega$ of ordinals is called a *downset*. The use of constructive ordinals, instead of the usual set-theoretic ordinals, provides canonical fundamental sequences (i.e., predecessors for limit ordinals) by construction, thus allowing for an elegant and robust formulation of ordinal recursion.

Now, we define the class $\mathcal{C}_\mathsf{A}$ of *predicative ordinal recursive functions on* A, where A is a downset of ordinals. We write $f(\bar{\mu}, \bar{n}; \bar{a})$ to refer to a function in the class. In this notation, before the semicolon we have inputs in normal position, of which $\bar{\mu}$ are ordinal inputs ranging over A and $\bar{n}$ range over numbers. After the semicolon, we have numeral inputs $\bar{a}$ in safe position. The output of any of these functions is a number.

Definition 1 (The class $\mathcal{C}_\mathsf{A}$). Let $\mathsf{A} \subseteq \Omega$ be a downset. We define $\mathcal{C}_\mathsf{A}$ as the smallest class of numeral functions with the inputs from A and $\mathbb{N}$ containing the number 0 as a nullary function, the projections on numeral inputs, the successor function $s(;a) = a+1$, the predecessor function (defined by $p(;0) = 0$, $p(;a+1) = a$), the conditional function (defined by $C(;a,b,c) = b$ if $a = 0$ and $C(;a,b,c) = c$, otherwise), and closed under the operations (1)–(4) listed below.

1. (predicative recursion) For $g, h_{\mathsf{suc}}, h_{\mathsf{lim}}, q(\bar{n};) \in \mathcal{C}_{\mathsf{A}}$, define f by
$$f(0,\bar{\nu},\bar{n};\bar{a}) = g(\bar{\nu},\bar{n};\bar{a})$$
$$f(\mu+1,\bar{\nu},\bar{n};\bar{a}) = h_{\mathsf{suc}}(\mu,\bar{\nu},\bar{n};f(\mu,\bar{\nu},\bar{n};\bar{a}),\bar{a})$$
$$f(\langle\mu_i\rangle_i,\bar{\nu},\bar{n};\bar{a}) = h_{\mathsf{lim}}(\langle\mu_i\rangle_i,\bar{\nu},\bar{n};f(\mu_{q(\bar{n};)},\bar{\nu},\bar{n};\bar{a}),\bar{a})$$
2. (safe composition) For $h, \bar{s}, \bar{t} \in \mathcal{C}_{\mathsf{A}}$, define the new function f by
$$f(\bar{\mu},\bar{n};\bar{a}) = h(\bar{\mu},\bar{s}(\bar{\mu},\bar{n};);\bar{t}(\bar{\mu},\bar{n};\bar{a})) \tag{1}$$
3. (constant substitution) For $g(\bar{\mu},\bar{n};\bar{a}) \in \mathcal{C}_{\mathsf{A}}$, define a new function by replacing one of the ordinal inputs of g with a fixed ordinal $\alpha \in \mathsf{A}$.
4. (structural rules) For $g(\bar{\mu},\bar{n};\bar{a}) \in \mathcal{C}_{\mathsf{A}}$, define a new function by performing either exchange, contraction or weakening on the ordinal inputs of g.

We denote $\mathcal{C}_{\mathsf{D}_\alpha}$ by $\mathcal{C}_\alpha$, for any $\alpha \in \Omega$.

Note that, in predicative recursion, $q(\bar{n};)$ selects the ordinal $\mu_{q(\bar{n};)}$ smaller than $\langle\mu_i\rangle_i$ that is used in the limit ordinal case, and that recursion happens in normal positions, with previous values applied in safe positions. Moreover, observe that composition with safe inputs is not allowed in normal position.

In order to assess the computational power of $\mathcal{C}_{\mathsf{A}}$, it is natural to relate this class to the known complexity classes of functions that are only defined over numeral inputs. For that reason, we restrict our focus to the functions in $\mathcal{C}_{\mathsf{A}}$ with numeral inputs only, which constitute the class $\mathrm{PredR}_{\mathsf{A}}$ formally defined as follows.

Definition 2 (The class $\mathrm{PredR}_{\mathsf{A}}$). For any downset $\mathsf{A} \subseteq \Omega$, let $\mathrm{PredR}_{\mathsf{A}}$ be the class of all functions $f : \mathbb{N}^k \to \mathbb{N}$ such that there is $\hat{f} \in \mathcal{C}_{\mathsf{A}}$ satisfying $f(\bar{n}) = \hat{f}(\bar{n};)$ for any $\bar{n} \in \mathbb{N}$. We denote $\mathrm{PredR}_{\mathsf{D}_\alpha}$ by PredR_α, for any $\alpha \in \Omega$.

Example 3. For a hint on the computational power of $\mathrm{PredR}_{\mathsf{A}}$, define by predicative recursion $G(0,n;) = 0$, $G(\mu+1,n;) = G(\mu,n;)+1$ and $G(\langle\mu_i\rangle,n;) = G(\mu_n,n;)$, and then define by constant substitution $G_\alpha(n;) = G(\alpha,n;)$ for each $\alpha \in \mathsf{A}$. This shows that the functions $\{G_\alpha\}_{\alpha\in\mathsf{A}}$—the *slow-growing hierarchy over* A—are all present in $\mathrm{PredR}_{\mathsf{A}}$.

To have a control on the ordinals used in the recursion, we define a family $\{\Phi_\alpha\}_{\alpha\leq\omega}$ of classes of ordinals where Φ_α consists of all constructive ordinals whose corresponding set-theoretic ordinal is strictly smaller than the Veblen ordinal $\phi_\alpha(0)$. Hence, the hierarchy $\{\Phi_\alpha\}_{\alpha\leq\omega}$ can be seen as a constructive counterpart of the Veblen hierarchy.

Definition 4. Let $\mathsf{A} \subseteq \Phi_\omega$ be a downset of ordinals. A is called *bounded* if there is $k \geq 1$ such that $\mathsf{A} \subseteq \Phi_k$. For any bounded A, if $\mathsf{A} \subseteq \mathsf{D}_{\omega^\omega}$, we define $l(\mathsf{A}) = 0$. Otherwise, define $l(\mathsf{A})$ as the least $k \geq 1$ such that $\mathsf{A} \subseteq \Phi_k$.

Our main contribution consists, then, of a complete characterization of predicative ordinal recursion for the constructive ordinals in Φ_ω (i.e., for ordinals up to $\phi_\omega(0)$).

Theorem 5 (Main Theorem). *If $\mathsf{A} \subseteq \Phi_\omega$ is a downset with $\mathsf{A} \not\subseteq \mathsf{D}_\omega$, then:*

- *(i) If A is bounded, then $\mathrm{PredR}_{\mathsf{A}} = \mathcal{E}_{l(\mathsf{A})+2}$.*
- *(ii) If A is not bounded, $\mathrm{PredR}_{\mathsf{A}}$ is the set of all primitive recursive functions.*

Corollary 6 (Main Corollary). $\mathcal{E}_2 = \mathrm{PredR}_{\omega^\omega}$, $\mathcal{E}_k = \mathrm{PredR}_{\Phi_{k-2}}$, *for any* $k \geq 3$, *and* $\mathrm{PredR}_{\Phi_\omega}$ *is the set of all primitive recursive functions.*

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