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We introduce the general concept of a non-wellfounded proof-system $\mathrm{G}(\mathcal{L})$, based on a non-wellfounded language $\mathcal{L}$, and analyze its properties through the lens of reverse mathematics, drawing connections to questions of determinacy. More concretely, we show that all of consistency, semantic cut-admissibility and soundness, for an appropriate semantic, for $\mathrm{G}(\mathcal{L})$ are equivalent to $\omega\,\mathrm{Det}(\mathcal{L})$. By the latter, we denote a principle stating that for countably many games based on the language $\mathcal{L}$, as explained in more detail below, there exists a set simultaneously containing a winning strategy for one of the two players for each of the countably many games. In most cases of interest, it is equivalent to the determinacy of the individual games, $\mathrm{Det}(\mathcal{L})$. Particular examples of this framework have been previously presented at CiE 2025 [1] and the ASL Logic Colloquium 2025, where we introduce systems $\mathrm{G}(\mathcal{L}_{\omega}^{\searrow P<\omega})$ and $\mathrm{G}(\mathcal{L}^{<\omega-\mathrm{trees}})$ of respective strengths $\Pi^1_1-\mathrm{CA}_0$ and projective determinacy PD.

Every element φ of a non-wellfounded language $\mathcal{L}$ has an associated ω-branching tree $T(\varphi)$, together with a labelling of its nodes by propositional connectives $c(\varphi) : T(\varphi) \to \{\vee, \wedge\}$, and a subset $\mathcal{W}(\varphi) \subseteq [T(\varphi)]$ of the infinite paths $[T(\varphi)]$ of $T(\varphi)$. Additionally, $\mathcal{L}$ carries a negation operation such that $T(\neg\varphi) = T(\varphi)$, but the labels $c(\neg\varphi)$ and winning paths $\mathcal{W}(\neg\varphi)$ are inverted. Defining sequents as finite lists $p_0, \dots, p_n$ of nodes $p_i \in T(\varphi_i)$ for some formulas $\varphi_i \in \mathcal{L}$, a notion of pre-proofs in the style of ω-logic readily follows from the propositional connectives $c(\varphi_i)$. Among those, the correct proofs are selected through a trace condition, asserting that for every branch B, there exists some $\varphi \in \mathcal{L}$ and a path $P \in \mathcal{W}(\varphi)$ such that P is fully unfolded along B. The connection to determinacy arises through an evaluation game for formulas φ. This is a two-player game, say between Player P and Opponent O, who collaboratively play a path through $T(\varphi)$. For $\vee$-nodes it is P's turn and for $\wedge$-nodes it is O's turn to choose the next child node. If one player is out of moves they lose, otherwise, if an infinite path $B \in T(\varphi)$ has been played, P wins iff $B \in \mathcal{W}(\varphi)$, otherwise O wins.

In the previously mentioned examples, $\mathcal{W}(\varphi)$ is determined by some additional structure that is continuously embedded along the paths of $T(\varphi)$. For $\mathcal{L}_{\omega}^{\searrow P<\omega}$, this additional data is a natural number label, called priority, which is non-increasing along branches. A path $P \in [T(\varphi)]$ is then winning, $P \in \mathcal{W}(\varphi)$, if the eventually constant label is even. For $\mathcal{L}^{1-\mathrm{trees}}$, we continuously embed another tree along the branches of $T(\varphi)$. This means that for every $s \in T(\varphi)$, we have an associated finite tree $T_1(s)$ such that if $s \sqsubseteq t$, then $T_1(s) \sqsubseteq T_1(t)$ is an initial segment. A path $P \in T(\varphi)$ is then winning, $P \in \mathcal{W}(\varphi)$, if the limiting tree $T_1(P) := \bigcup_{s \in P} T_1(s)$ is ill-founded. The language $\mathcal{L}^{<\omega-\mathrm{trees}}$ is obtained by a higher-order generalization of this idea.

Finally, if time permits, we comment on syntactic cut-elimination for the systems $\mathrm{G}(\mathcal{L})$. We show for some general classes of non-wellfounded languages, including the previous examples, that correctness of syntactic cut-elimination is once again equivalent to $\omega\,\mathrm{Det}(\mathcal{L})$. This basic idea follows an approach by Baelde et. al. [2], reducing syntactic cut-elimination to soundness of the system.

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[1] Provenzano, Philipp, *On the reverse mathematics of cut-elimination and determinacy*, ***Lecture Notes in Computer Science*** (Lisbon, Portugal), (Beckmann, Arnold and Oitavem, Isabel), Springer-Verlag, 2025, to be published, accepted for Computability in Europe 2025

[2] Baelde, David and Doumane, Amina and Saurin, Alexis, *Infinitary Proof Theory: the Multiplicative Additive Case*, ***Leibniz International Proceedings in***

Informatics (LIPIcs) (25th EACSL Annual Conference on Computer Science Logic), (Talbot, Jean-Marc and Regnier, Laurent), vol. 62, 2016, pp. 42:1–42:17.