▸ JUSTUS BECKER, *A non-wellfounded and labelled sequent calculus for bimodal provability logic*.
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The notion of *provability logic* stems from the idea of interpreting the $\Box$ in modal logic as a provability operator in a sufficiently strong theory. The most well known provability logic is the modal logic Gödel-Löb (GL), for which it was shown in [25] by Robert Solovay that it is complete for the provability of Peano Arithmetic (PA). However, we can find many more provability logics that extend GL by considering the provability of a theory T over a different metatheory U. This notion was independently introduced by Sergei Artemov [2] and Albert Visser [27]. Similarly, we might consider tuples of different theories, and interpret their provability as a tuple of modalities, giving us multi-modal provability logics which have been first studied by Craig Smoryński [24] and Tim Carlson [8]. The easiest class of such multi-modal provability logics are *bimodal provability logics*, where we consider the provability of two theories T, U under a common metatheory $T \cap U$. Note that one might consider variant notions of bimodal provability logic, such as a different metatheory or different kinds of provability predicates (see e.g. [12, 28]).

While a lot has been done in the field of bimodal provability logics, such as the study and classification of different logics, as well as their semantics and decidability [6, 5, 29, 30], there does not exist any analytic proof system for them thus far, making the following system the first proper calculus for bimodal provability logic. Although the calculus we introduce is new, we build upon the existing literature of proof theory for the provability logic GL and modal logic in general; most notably we adapt the system for classical GL from [9] into a bimodal setting.

The first technique we employ are *labelled systems*, which can be traced back to [14], where Stir Kanger provided a cut-free system for $\mathsf{S5}$ (he talks about *spotted* formulas instead of labelled ones); however, labelled calculi have been popularised only later in broader studies of non-classical logics (see e.g. [11, 17, 26]). Labels allow us to internalise the structure of semantics into the calculus. In our setting, we use *labelled formulas*, written as $x : A$, and two kinds of *relational atoms*, xRy and xSy. Labelled sequents then have the form $\mathcal{R}, \Gamma \Rightarrow \Omega$, where $\mathcal{R}$ is a set of relational atoms, while Γ and Ω are sets of labelled formulas. Relational atoms can be seen as a Kripke frame whereas labelled formulas determine truth and falsity, making it a model. This not only gives us easy semantic completeness but also allows for a modular adaptation of the system towards extensions.

The second proof theoretic technique we use relies on *non-wellfounded proofs*, which are a generalisation of cyclic proofs. Cyclic proof systems were originally developed for the modal μ-calculus (see [18]) where the circularity has been used to encapsulate the meaning of the fixpoint operators. The recursiveness that can be captured by cyclic proofs has also been adapted for induction [7, 23], propositional dynamic logic [10], and other modal fixpoint logics, such as GL and Grz [16, 20, 21]. The latter of these is interesting for our purposes as we build on the results of non-wellfounded and cyclic proofs for GL.

We adapt the labelled system $\ell\mathsf{GL}$ from [9] into a bimodal setting. The general idea of $\ell\mathsf{GL}$ is that the full Kripke semantics for GL is internalised. While it is easy to do that for the first-order condition of transitivity, the second-order property of converse-wellfoundedness is internalised by the *progress condition* on non-wellfounded proofs. Thus we enhance the semantic ideas of labels by essentially reasoning about infinite models. More precisely, proof search should fail if and only if we can obtain a countermodel from the failed proof. Usually such a countermodel can be extracted from a branch of a proof tree that is not closed: either that branch got stuck and no rule can be applied to its leaf, or the branch continues indefinitely. Consider now a proof with an infinite branch such that the countermodel extracted from it contains an infinite chain. Such a countermodel, however, is not a countermodel for GL. We therefore call such a branch *progressing*, and allow progressing branches to occur in a valid proof of ℓGL.

We call our system labCS$^\infty$ (labelled CS with infinite branches) for which we show that it is sound and complete for the basic bimodal provability logic CS. The logic is basic insofar as it is a sublogic of any bimodal provability logic (according to the description at the beginning). As we model our system after the semantics of CS, the proof of both soundness and completeness is also done via models. Here, we rely on interpreting sequents (not only formulas) in models. Soundness is done via local (semantic) soundness of the rules as well as soundness of the progress condition. Completeness is shown by a countermodel extraction from failed proofs.

For ongoing research, we conjecture that the system labCS$^\infty$ can be used as a tool towards describing other bimodal provability logics. This should be simple for Kripke-frame complete logics with a first-order correspondence, such as CSM and P, as we can write the first-order conditions as additional rules (see [4, Chapter 8] for an overview of bimodal provability logics where these logics are also defined). The bimodal provability logic of PA and Zermelo-Fraenkel set theory (ZF), called ER (*essentially reflexive*), is Kripke-frame incomplete and thus poses an interesting obstacle for our setup. However, there exists a generalised Kripke semantics for this logic due to Albert Visser [29]. One possible way of describing the admissible valuation sets in these models for ER is by an infinite descent. It therefore seems natural that we can utilise the non-wellfoundedness already present in labCS$^\infty$ to obtain a system for the logic ER by adding an additional progress condition. The idea is similar to before: an infinite progressing branch should not give us a countermodel for the logic. Thus, any model which we read out from a progressing branch has to have a valuation which is not an admissible set in the generalised semantics.

Lastly, we want to highlight further directions on how to use labCS$^\infty$ as a basis for other sequent calculi. This might include other multi-modal provability logics such as GLP [13] or GR [22], as well as non-normal provability logics such as GLS [25, §5] and GL_ω [3]. We might also translate some of these systems into sequent systems without structure (i.e. without labels, nestings or anything alike). This can be done by *sequentialising* the proofs by first transforming them into a normal form, similar to [15, 19]. This might allow us to gain a pure sequent system which, due to its construction, is immediately sound and complete. The reduced structure of such a system might then allow for easier proof theoretic investigations such as proving properties like interpolation (see e.g. [1, 21]).

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