▸ VICTOR BARROSO-NASCIMENTO, MARIA OSÓRIO COSTA, AND ELAINE PIMENTEL, *Bilateralist base-extension semantics with incompatible proofs and refutations.*
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The traditional Fregean account of negation depicts denial of a proposition as the mere assertion of its opposite [6]. As Frege himself puts it:

> To each thought there corresponds an opposite, so that rejecting one of them coincides with accepting the other. To make a judgement is to make a choice between opposite thoughts. Accepting one of them and rejecting the other is one act. So there is no need of a special sign for rejecting a thought. We only need a special sign for negation as such. [4, pg. 198]

Logic has traditionally embraced the Fregean view, dealing with rejection only indirectly through definition of assertion conditions for negation.

This view has been challenged by contemporary logicians, leading to the creation of what is now known as logical bilateralism [10, 8]. According to bilateralists [17, 18], assertion and rejection are distinct speech acts that cannot be reduced to each other, so a proper account of logic should distinguish between conditions for propositional assertion and for propositional rejection [10]. Bilateralism was originally proposed as a justification of classical logic, but it is arguably tenable even from a intuitionistic viewpoint [7]. Explicit accounts of rejection allow the definition of natural deduction systems for classical logic that, in virtue of their harmonic rules, enjoy proof-theoretic properties previously ascribed only to intuitionistic natural deduction [10, 3].

From a proof-theoretic perspective, the distinction between assertion and rejection is important inasmuch it leads to a stronger distinction between *proofs* and *refutations*. Proofs are traditionally conceived as structures capable of guaranteeing epistemic grounds for assertion, and refutations as structures that do the same but for rejection. If rejections are nothing more than negative assertions then refutations are nothing more than negative proofs, but as soon as rejection emancipates itself from assertion the possibility of formulating an independent concept of refutation presents itself. This insight is especially important in the intuitionistic setting, which often takes the concept of proof to be one of its central philosophical notions [15, 14, 16]. If refutations are independent from proofs and intuitionistic logic only deals with proofs, the natural conclusion would be that it is an incomplete (or at least partial) logic. This claim is reinforced by the observation that development of the intuitionistically acceptable co-implication operator, studied extensively in bi-intuitionistic logics [9, 5], is historically tied to the development of constructive notions of duality [2, 19]. To illustrate, the refutation rules for the co-implication in the natural deduction system $\mathsf{2Int}$ [17, 1] are:

$$\begin{array}{c} \Gamma; \Delta, [\![\psi]\!] \\ \Pi \\ \hline\hline \phi \\ \hline\hline \phi \mapsfrom \psi \end{array} \; I \mapsfrom (-) \qquad\qquad \begin{array}{cc} \Gamma_1; \Delta_1 & \Gamma_2; \Delta_2 \\ \Pi_1 & \Pi_2 \\ \hline\hline \phi \mapsfrom \psi & \psi \\ \hline\hline \multicolumn{2}{c}{\phi} \end{array} \; E \mapsfrom (-)$$

$\mathsf{2Int}$' rules distinguish between proof rules and refutation rules, as well as between proof assumptions (or simply assumptions) and refutation assumptions (or contra-assumptions). The dependencies of a deduction are represented by a pair $\Gamma; \Delta$, in which Γ is a set of proof assumptions and Δ a set of refutation assumptions. Proof rules are denoted by a single line; refutations rules by a double line.

Unfortunately, in 2Int–as well as in all other bilateral systems known to the authors–the simultaneous derivation of a formula and its refutation does not lead to contradiction.[1]

In this work, we introduce a new bilateral system, 2IntPR, built upon 2Int, in which the joint derivability of a formula and its refutation does lead to contradiction by adding the following rules:

$$\dfrac{\overline{A} \qquad \overline{\overline{A}}}{\bot}\, PR(+) \qquad\qquad \dfrac{\overline{A} \qquad \overline{\overline{A}}}{\top}\, PR(-)$$

We show that 2IntPR enjoys desirable proof-theoretic properties, including normalization.

We then define a base-extension semantics for this system, allowing atomic bases to include both atomic proof and atomic refutation rules. Base-extension semantics [11] (BeS) is a strand of Proof-theoretic semantics [12, 13] (PtS) where proof-theoretic validity is defined relative to a given collection of inference rules regarding basic formulas of the language. More specifically, in BeS the characterisation of consequence is given by an inductively defined semantic judgment (*support*) whose base case is given by provability in an *atomic system* (or a *base*), where a base is a collection of rules involving only "basic" formulas–in general atoms and possibly the falsity unity $\bot$. In the particular case of 2IntPR, the support relations $\Vdash^+_{\mathcal{B}}, \Vdash^-_{\mathcal{B}}, \Vdash^+$ *and* $\Vdash^-$ for the atomic case are defined as follows:

(At+) $\Vdash^+_{\mathcal{B}} p$ iff $\vdash^+_{\mathcal{B}} p$, for $p \in$ At;

(At−) $\Vdash^-_{\mathcal{B}} p$ iff $\vdash^-_{\mathcal{B}} p$, for $p \in$ At;

The remaining support clauses are defined inductively. We prove that our proposed semantics is sound and complete with respect to the bilateral calculus 2IntPR.

This framework also enables us to establish the following key results:

- The system validates all of the following:
  - $\bot, \Gamma; \Delta \Vdash^+_{S} A$
  - $\bot, \Gamma; \Delta \Vdash^-_{S} A$
  - $\Gamma; \Delta, \top \Vdash^+_{S} A$
  - $\Gamma; \Delta, \top \Vdash^-_{S} A$

  These clauses, notably, do not hold in the original semantics for 2Int.
- Mutual exclusivity between assertion and denial: If $\Vdash^+ \mathcal{B}A$, then $\nVdash^- \mathcal{B}A$, and conversely, if $\Vdash^- \mathcal{B}A$, then $\nVdash^+ \mathcal{B}A$.

Finally, within this system, we can show that a refutation of A constructively yields a proof of $\neg A$–but not the other way around. This is a noteworthy feature: it reflects the fact that an explicit construction of a counterexample (which our system interprets as a form of refutation) naturally supports the negation of a statement. After all, such a counterexample can be used to define a procedure that derives a contradiction from assuming A. However, the existence of a procedure that derives a contradiction from A–*i.e.* a proof of $\neg A$–does not necessarily provide a concrete counterexample. This asymmetry explains why, in our system, negation does not entail rejection.

[1] In particular, for the reader acquained with Base Extension Semantics, if a base is explosive with respect to proofs and refutations, then we can prove and refute all formulas!

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