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General inductive definitions and their proof theory from a perspective of knowledge representation.

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Inductive definitions are ubiquitous in mathematics and computer science, and play an important role in knowledge representation. The logic FO(ID) is an extension of classical first-order logic that serves as a formal scientific theory of inductive definitions as they occur in mathematical texts [1, 2] . Inspired by logic programming, an inductive definition is represented in FO(ID) as a set Φ of *definitional rules*. These are expressions of the form $\forall \bar{x} : P(\bar{t}) \leftarrow \varphi$, where $\bar{x}$ is a tuple of variables, $P(\bar{t})$ an atom, and φ a first-order formula. Definitions are assigned meaning in FO(ID) through the *well-founded semantics*. This semantics was originally developed for logic programs [3], but Denecker et al. have demonstrated its close connection to inductive definitions [4, 5, 1, 6]. They argued that the well-founded semantics formalize the construction processes behind inductive definitions.

FO(ID) has several advantages over alternative logics of inductive definitions: 1) It has a very general notion of inductive definition, as it does not impose syntactic restrictions on its rules such as *positivity* [7, 8, 9, 10] or *stratification* [11, 12, 13, 14]. A *positive* definition is one without negation, whereas a stratified definition is one in which negation may only occur in a well-behaved, 'layered' way. While entailing convenient properties, these syntactic constraints exclude some natural and important examples of inductive definitions. By removing these syntactic constraints, FO(ID) supports a broad class of inductive definitions, ranging from monotone inductive definitions to inductive definitions over a well-founded order [1]. 2) FO(ID) supports *non-total* definitions, for which the construction process does not end with a well-defined object. These definitions can intuitively be seen as *flawed* definitions. For instance, $\{P \leftarrow \neg P\}$ is a simple example of a non-total definition. Allowing non-total definitions may seem bizarre at first, as definitions in (good) mathematical texts are always total. However, natural language permits non-total definitions, e.g.,"let x be the smallest number that cannot be described in less than 20 words". The definition $\{P \leftarrow \neg P\}$ can be seen as a formalization of a well-known philosophical paradox called the *liar paradox*, which says: "this sentence is false". By allowing non-total definitions, the classical logic FO(ID) offers a theoretical framework to separate 'good' (i.e., total) from 'bad' (i.e., non-total) definitions, which is interesting from a linguistic and a philosophical point of view. 3) FO(ID) provides a meta-perspective on definitions by explicitly incorporating definitions in its language. This enables its users to talk and reason about definitions *themselves* instead of only the objects they define. For instance, the formula $\Phi_1 \Leftrightarrow \Phi_2$ expresses the equivalence of the definitions Φ_1 and Φ_2, and $\neg\Phi$ expresses that the definition Φ is non-total.

Most proof systems that have been developed for logics of inductive definitions restrict to positive definitions [15] or stratified definitions [13, 16, 17]. The exception to this rule is the work of Hou et al., who developed a sequent calculus for the propositional fragment of FO(ID) [18], as well as a sequent calculus for an extension of FO(ID) with stratified least fixpoint expressions [19]. However, the latter is unsatisfactory for at least two reasons: 1) The inference rules are rather complicated, and one rule even has infinitely many premises. 2) The inference rules can be seen as unnatural, as they use the intricate well-founded semantics to prove theorems about inductively defined predicates. However, mathematical proofs about inductively defined predicates typically use different techniques, such as mathematical induction or infinite descent.

These issues are circumvented by the sequent calculus LFO(ID) for FO(ID) [20], which is based on the more familiar principle of mathematical induction. LFO(ID) can be seen as an extension of the sequent calculus LKID by Brotherston and Simpson [15] to a much broader class of inductive definitions. While LKID covers only positive definitions, LFO(ID) supports formal proofs about non-positive, non-stratified and even non-total definitions. As an interesting side feature, LFO(ID) also permits proofs of non-totality of definitions, thereby providing a proof-theoretical tool to show that a given definition is flawed.

This talk will be about the logic FO(ID) and the corresponding sequent calculus LFO(ID).

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