DEFINITENESS PROPERTIES OF ARITHMETICAL THEORIES

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The talk will present a few fairly recent results about four properties that express forms of internal completeness or categoricity of sequential theories: tightness, semantic tightness, neatness, and solidity.

We say that interpretations $\sigma_1, \sigma_2 \colon S \lhd T$ are isomorphic if there is a formula that acts in T as an isomorphism between the structures given by σ_1 and σ_2. Theories S and T are bi-interpretable via $\sigma \colon S \lhd T$ and $\tau \colon T \lhd S$ if $\tau \circ \sigma \simeq \mathrm{id}_T$ and $\sigma \circ \tau \simeq \mathrm{id}_S$. A theory T is tight if any two bi-interpretable $\mathcal{L}_T$-theories containing T are equal. Tightness can be thought of as a form of internal or local completeness of a theory: a tight theory has no two distinct extensions that can 'see each other', in the sense of bi-interpretability. Semantic tightness is a stronger, semantic analog of tightness; it is defined in terms of interpretations between models and can be thought of as a form of internal categoricity. Neatness and solidity are stronger versions of tightness and semantic tightness respectively, defined using interpretability retracts instead of bi-interpretations.

Partly building on the work of Visser [1], Enayat proved that PA, Z_2, ZF, and KM have all of the above properties [2]. On the other hand, it was shown later by Enayat, Freire, Hamkins, Łełyk, and Williams that a number of natural subtheories of the theories mentioned above, including all of their finitely axiomatizable fragments, do not have any of these properties [2, 3, 4, 5]. These authors posed variants of a natural question: 'Is there a proper subtheory of PA, Z_2, ZF, or KM that has any of these properties?'. During the talk, I will answer this question in the cases of PA and Z_2. I will also briefly discuss examples of theories that separate the properties mentioned above.

Most of the talk will be based on joint work with Leszek Kołodziejczyk and Mateusz Łełyk.

References

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