▶ JULIEN DAOUD AND FEDOR PAKHOMOV, *Speedup for Presburger arithmetic and real closed fields*.
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The investigation of speedup phenomenon for first-order theories, i.e. situations where for some sentences provable in both theories T and U their proofs in T are much shorter than in U, goes back to the classical result of Gödel [1]. Typically the speedup phenomenon for first order theories has been studied for sufficiently expressive theories that can interpret arithmetic over natural numbers with addition and multiplication. In this setting we observe the following empirical phenomenon: either the length of proofs in theories are equivalent upto polynomial transformation, or a (stronger) theory has at least hyper-exponential speedup over a (weaker) theory.[1]

In this talk we consider Presburger arithmetic PrA and the theory of real closed fields RCF. In our setting we provide natural examples that the aforementioned empirical phenomenon does not hold: examples where the stronger theory has super-exponential speedup over the weaker theory but where the speedup is bounded from above by a finite tower of exponentiations, i.e. less than hyper-exponential. Notice that the sub-hyper-exponential upper bounds for the speedups essentially come from the known results about the complexity theoretic upper bounds for the decision problems for the investigated theories. Hence we are left to prove the lower bound.

Due to quantifier elimination in these theories (see [2] and [4]), there are two kinds of natural ways to axiomatize them. Namely, on one hand PrA can be axiomatized with the full schemata of first-order induction, denoted by PrA, and RCF with the full schemata of first-order least upper bound principle, denoted by $Tarski$. While on the other hand there are rather natural axiomatizations of these theories that avoid the use of formulas of unbounded quantifier depth; we denote those axiomatizations respectively by PrA_{alt} and RCF.

In this talk we compare these two groups of axiomatizations from the perspective of the lengths of proofs. We show that the first group of axiomatizations enjoys at least double exponential speedup. On the technical level the main idea of our proofs is to construct short sentences expressing the validity of much longer axioms from the less efficient axiomatizations in such a way that the produced sentences have short proofs in the more efficient axiomatization.

The main theorems that we prove are:

THEOREM 1. *There is a* $2^{2^{x^{\varepsilon}}}$ *speedup of* PrA *over* PrA_{alt}.

THEOREM 2. *There is a* $2^{2^{x^{\varepsilon}}}$ *speedup of* $Tarski$ *over* RCF.

[1] KURT GÖDEL, *Über die Länge von Beweisen*, ***Ergebnisse eines mathematischen Kolloquiums***, vol. 7 (1936), pp. 23–24.

[2] MOJZESZ PRESBURGER, *Uber die Vollstandigkeiteines gewissen Systems der Arithmetik ganzer Zahlen, in welchen die Addition als einzige Operation hervortritt*, ***Comptes-rendus du ler congres des mathematiciens des pays slavs***, 1929.

[3] PAVEL PUDLÁK, *The lengths of proofs*, ***Studies in Logic and the Foundations of Mathematics*** vol. 137, Elsevier, 1998, pp. 547–637.

[4] ALFRED TARSKI AND J.C.C. MCKINSEY, *A Decision Method for Elementary Algebra and Geometry*, University of California Press, 1951.

[1] See for example classical results in [3].