▶ MARCEL ERTEL, EDUARDO SKAPINAKIS, *Non-deterministic operations in applicative theories.*
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We use applicative theories (see [2]) to give machine-independent characterizations of classes of computational complexity. Strahm [5, 6] introduced the theory PT, whose class of provably total functions coincides with the class of functions computable in polynomial time. We strengthen PT with axioms of *tree induction*, inspired by a recursion scheme of Oitavem [3, 4], which directly represents the behavior of a non-deterministic Turing machine, and give a new proof theoretic characterization of the polynomial time hierarchy of functions and of each of its levels. Moreover, we show that our approach is, essentially, unavoidable, as no class between `FPTime` and `FPSpace` can be obtained by simpler forms of tree induction.

Definition 1. $\texttt{FPTime}^{\mathcal{D}}$ is the class of functions computable by a deterministic Turing machine in polynomial time with an oracle for $\mathcal{D}$. $\texttt{NPTime}^{\mathcal{D}}$ the class of sets decidable by a non-deterministic Turing machine in polynomial time with an oracle for $\mathcal{D}$. The polynomial time hierarchy is defined as $\texttt{PH} := \cup_{k\geq 0}\Sigma^p_k$, where: (i) $\Sigma^p_0 := \texttt{P}$; (ii) $\Sigma^p_{k+1} := \texttt{NPTime}^{\Sigma^p_k}$. For $k \geq 0$, define $\Box^p_{k+1} := \texttt{FPTime}^{\Sigma^p_k}$ and $\texttt{FPH} := \cup_{k\geq 1}\Box^p_k$.

Definition 2. The language $\mathcal{L}$ is defined as follows:

$$\text{(terms) } t := \mathcal{X} \,|\, \mathsf{k} \,|\, \mathsf{s} \,|\, \epsilon \,|\, \mathsf{s}_0 \,|\, \mathsf{s}_1 \,|\, \mathsf{p_W} \,|\, * \,|\, \times \,|\, \mathsf{c_W} \,|\, \mathsf{c}_{\leq} \,|\, (t \cdot t)$$

$$\text{(formulas) } \phi := \mathsf{W}(t) \,|\, t = t \,|\, \neg\phi \,|\, (\phi \wedge \phi) \,|\, (\phi \vee \phi) \,|\, (\phi \to \phi) \,|\, (\exists x)\phi \,|\, (\forall x)\phi$$

where $x \in \mathcal{X}$, for $\mathcal{X} = \{x, y, z, f, g, h, \dots\}$ a set of variables (possibly containing subscripts).

The constants represent the "initial functions" of a function algebra [1]. The binary function symbol $\cdot$ represents term application. The predicate W stands for "being a binary word".

Definition 3. B is the theory defined over $\mathcal{L}$, based on classical logic with equality, containing further the obvious axioms characterizing the combinators k and s and the other non-logical symbols, as well as an axiom of extensionality.

Definition 4. A function $F : \mathbb{W}^n \to \mathbb{W}$ is *provably total* in an applicative theory T extending B if there exists a closed $\mathcal{L}$ term t such that:

(i) for each $\sigma_1, \dots, \sigma_n \in \mathbb{W}$, $\mathsf{T} \vdash t\ \overline{\sigma_1} \dots \overline{\sigma_n} = \overline{F(\sigma_1, \dots, \sigma_n)}$
(ii) $\mathsf{T} \vdash t : \mathsf{W}^n \to \mathsf{W}$

The set of such functions is denoted by $\mathsf{Comp}(\mathsf{T})$.

Definition 5 ([6]). $\Sigma^{\mathsf{b}}_{\mathsf{W}}$ denotes the class of formulas of the form $A(y) := (\exists z \leq ty)\, B(t, y, z)$ for $B(t, y, z)$ a positive[1] and W-free formula. The term t is called the *bounding term*.

Definition 6 ([6]). Denote by ($\Sigma^{\mathsf{b}}_{\mathsf{W}}$-$\mathsf{I_W}$) the scheme that includes, for every $\Sigma^{\mathsf{b}}_{\mathsf{W}}$ formula A, with bounding term t, the following axiom:

$$\Big(t : \mathsf{W} \to \mathsf{W} \ \wedge\ A(\epsilon) \ \wedge\ (\forall y \in \mathsf{W})\Big[A(y) \to \Big(A(y0) \wedge A(y1)\Big)\Big] \Big) \to (\forall y \in \mathsf{W})\, A(y) \qquad (\Sigma^{\mathsf{b}}_{\mathsf{W}}\text{-}\mathsf{I_W})$$

[1] A formula is positive if it does not contain negations nor implications.

Proposition 7 ([6, Theorem 23])**.** Define $\mathsf{PT} := \mathsf{B} + (\Sigma^{\mathsf{b}}_{\mathsf{W}}\text{-}\mathsf{I}_{\mathsf{W}})$, i.e., as the theory extending B with the axiom scheme $(\Sigma^{\mathsf{b}}_{\mathsf{W}}\text{-}\mathsf{I}_{\mathsf{W}})$. $\mathsf{Comp}(\mathsf{PT}) = \mathtt{FPTime}$.

Definition 8. For Ξ a class of formulas, denote by $(\Xi\text{ -}\mathsf{TRI}^{\vee}_{\mathsf{W}})$ the scheme of disjunctive tree induction on W, which includes, for every formula $A \in \Xi$, the following axiom:[2]

$$\Bigg((\forall p \in \mathsf{W})\Big(A(\epsilon, p, 0) \vee A(\epsilon, p, 1)\Big) \;\wedge$$

$$(\forall z, w)(\forall y, p \in \mathsf{W})\Big[\Big(A(y, p0, z) \wedge A(y, p1, w)\Big) \to \Big(A(y0, p, z \,\dot{\vee}\, w) \wedge A(y1, p, z \,\dot{\vee}\, w)\Big)\Big]\Bigg)$$

$$\to (\forall y, p \in \mathsf{W})\Big(A(y, p, 0) \vee A(y, p, 1)\Big) \qquad (\Xi\text{ -}\mathsf{TRI}^{\vee}_{\mathsf{W}})$$

Definition 9. Let $\mathcal{E}$ (easy) stand for the class of formulas which are positive, do not contain the predicate W, and do not contain disjunctions.

We define the following applicative theory.

Definition 10. $\mathsf{PHT} := \mathsf{B} + (\Sigma^{\mathsf{b}}_{\mathsf{W}}\text{-}\mathsf{I}_{\mathsf{W}}) + (\mathcal{E}\text{ -}\mathsf{TRI}^{\vee}_{\mathsf{W}})$

We further define a hierarchy of theories, by restricting the number of times that tree induction can be used in a proof.

Definition 11. For each $k \geq 0$, $\mathsf{PHT}^{\mathsf{k}}$ is the restriction of PHT, where every proof uses at most k times the induction scheme $(\mathcal{E}\text{ -}\mathsf{TRI}^{\vee}_{\mathsf{W}})$.

We establish the following characterizations:

Theorem 12. For each $k \geq 0$, $\mathsf{Comp}(\mathsf{PHT}^{\mathsf{k}}) = \square^{p}_{k+1}$.

Theorem 13. $\mathsf{Comp}(\mathsf{PHT}) = \mathtt{FPH}$.

Of independent interest, we also make two further contributions: a new method for proving lower bounds in applicative theories, based on the notion of *Turing completeness*, and a dichotomy result for tree induction, which gives necessary conditions for similar approaches to characterize classes between `FPTime` and `FPSpace`.

[1] Clote, Peter. Computation models and function algebras. Studies in Logic and the Foundations of Mathematics, 140:589–681, 1999.

[2] Jäger, Gerhard and Kahle, Reinhard and Strahm, Thomas. On Applicative Theories. Logic and Foundations of Mathematics, 83–92, 1999.

[3] Oitavem, Isabel. A recursion-theoretic approach to NP. Annals of Pure and Applied Logic 162(8):661–666, 2011.

[4] Oitavem, Isabel. The polynomial hierarchy of functions and its levels. Theoretical Computer Science, 900: 25–34, 2022.

[5] Strahm, Thomas. Polynomial time operations in explicit mathematics. The Journal of Symbolic Logic, 62(2): 575–594, 1997.

[6] Strahm, Thomas. Theories with self-application and computational complexity. Information and Computation, 185(2): 263–297, 2003.

[2] The term $\dot{\vee}$ returns 1 if and only if one of its inputs is a word ending in 1.