▸ IAN SHILLITO, *Towards computing uniform interpolants for KM in Rocq.*
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Background. Kuznetsov-Muravitsky logic (KM) is an intuitionistic modal logic created in the late 1970s expressed in the following language: $\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi$. The interest in KM lies in its deep connection to (propositional) intuitionistic logic IL and provability logic GL: all non-modal extensions of KM are conservative extensions of their corresponding intermediate logics, and the lattice of extensions of KM is isomorphic to the one of GL. Axiomatically, this logic is an extension of the basic intuitionistic modal logic iK with the strong Löb axiom $(\Box\varphi \to \varphi) \to \varphi$, characterising the logic iSL, and $\Box\varphi \to (\psi \vee (\psi \to \varphi))$, denoted $\mathbb{KM}$. A Kripke semantics for KM exists, where the intuitionistic relation $\leq$ is reflexive and transitive, while the modal relation R is transitive, converse well-founded, and corresponds exactly to the irreflexive part of $\leq$, i.e. $R =<$. For further details about KM, see [16].

Uniform interpolation is a strong form of interpolation, saying that *propositional* quantifiers can be defined inside the logic. The left (resp. right) uniform interpolant of a formula φ with respect to a variable p is a p-free formula, denoted $\forall p\varphi$ (resp. $\exists p\varphi$), such that for any p-free ψ:

$$\vdash \forall p\varphi \to \varphi \qquad \frac{\vdash \psi \to \varphi}{\vdash \psi \to \forall p\varphi} \qquad \vdash \varphi \to \exists p\varphi \qquad \frac{\vdash \varphi \to \psi}{\vdash \exists p\varphi \to \psi}$$

A logic has uniform interpolation if both these interpolants exist for any formula. This property is notoriously hard to prove, but techniques do exist: through the Kripke semantics using specific classes of finite frames [19, 20], the algebraic semantics via the notion of *coherence* [15], and proof theory by involving extremely well-behaved calculi. Here we focus on the last kind of technique and make progress in its adaptation to KM, in the hope of eventually obtaining the novel result that KM has uniform interpolation.

The proof-theoretic technique to prove uniform interpolation is due to Pitts [17], who used it on IL. At a high level, Pitts' recipe is quite simple: take a cut admitting sequent calculus which strongly terminates (a measure on sequents strictly decreases upwards in its rules), and compute the interpolant of a sequent recursively on its (finite) proof-search tree. To date, this technique was refined [5, 6] and reused in a variety of places both for logics on an intuitionistic basis [8, 7] and for logics on a classical basis [1, 7]. In fact, some of these results were fully formalised in the interactive theorem prover Rocq [5, 7, 6], notably those pertaining to the intuitionistic modal logic iSL.

A cousin of KM. As mentioned above, iSL was recently shown to have uniform interpolation [7]. This proof à la Pitts relied on a strongly terminating calculus $\mathsf{G4iSLt}$ for the logic developed the year before [18, 9]. Given the intuitionistic nature of iSL, the design of $\mathsf{G4iSLt}$ amounted to the expansion of the strongly terminating calculus $\mathsf{G4ip}$ [3, 12] for IL to the modality of iSL. As is well known, the key idea behind $\mathsf{G4ip}$ is to break down the usual implication left rule, which contains an explicit contraction from conclusion to (left) premise on the principal formula $\varphi \to \psi$, into several rules: one for each connective, which could be the main one of φ. Therefore, to treat iSL we not only add to $\mathsf{G4ip}$ its rule for $\Box$ but also an implication left rule where φ is a boxed formula, thereby following Iemhoff's work on $\mathsf{G4i}$ calculi for intuitionistic modal logics [13, 14]. In these rules, presented below, $\otimes$ is an operation on multisets preserving all non-boxed formulas and unboxing the boxed ones.

$$\frac{\otimes\Gamma, \Box\varphi \Rightarrow \varphi}{\Gamma \Rightarrow \Box\varphi}\ \Box\mathrm{R} \qquad\qquad \frac{\otimes\Gamma, \psi, \Box\varphi \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow \chi}{\Gamma, \Box\varphi \to \psi \Rightarrow \chi}\ \Box\!\to\! L$$

Note that the rule $(\Box \to L)$ hides a local application of *modus ponens*, as ψ in the premise is generated by the *diagonal formula* $\Box\varphi$, characteristic of provability logics, and the principal formula $\Box\varphi \to \psi$. The calculus $\mathsf{G4iSLt}$ can be shown to be terminating in different ways, either using a measure inherited from GL [10, 11] counting the boxed formula used in the proof search, or a simpler and more elegant one designed by Férée [7] and extending Pitts'.

G4KM. Given the proximity of iSL and KM, one could hope to obtain a strongly terminating calculus for the latter by modifying G4iSLt. Crucially, the modified calculus must prove $\mathbb{KM}$.

When trying to prove the sequent $\Box\varphi \Rightarrow \psi \vee (\psi \to \varphi)$ we face a first obstacle: the only G4iSLt rule applicable is the disjunction right rule, forcing us to either pick ψ or $\psi \to \varphi$. Obviously, neither of these choices leads to a proof: only preserving both disjuncts can lead to one. Therefore, a proof for this axiom calls for *multi-succedent* sequents, giving a better suited disjunction on the right rule. However, to stay sound we are compelled to use a (in-)famous implication right rule.

$$\frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ {\scriptstyle \vee R} \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ {\scriptstyle \to R}$$

We are not out of the woods yet: proving $\Box\varphi \Rightarrow \psi, \psi \to \varphi$ still looks impossible, as if we were to use $(\to R)$ we would lose ψ on the way, and thereby the possibility to prove the sequent. This suggests that the rule $(\to R)$ is not the adequate one. By closely inspecting the Kripke semantics, one can arrive at the following alternative rule.

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta \quad \otimes\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ {\scriptstyle \to R'}$$

In essence, this rule performs a *case analysis* in the Kripke semantics. Read from bottom to top, and interpreting being on the left (resp. right) of the sequent as meaning being satisfied (resp. falsified), the conclusion tells us that the implication $\varphi \to \psi$ is falsified at a world w. For this to happen, there must be a $v \geq w$ satisfying φ but falsifying ψ. This v can be w itself, as $\leq$ is reflexive, giving us the left premise. Else, v has to be such that $w < v$, forcing the deletion of Δ as falsity is not persistent, and letting us unbox the boxed formula in Γ, whence $\otimes\Gamma$, as we effectively performed a modal jump: remember, R is exactly $<$! We obtained the right premise.

A proof of $\Box\varphi \Rightarrow \psi, \psi \to \varphi$ is then straightforward.

$$\frac{\dfrac{}{\Box\varphi, \psi \Rightarrow \psi, \varphi}\ {\scriptstyle \mathrm{Id}} \qquad \dfrac{}{\varphi, \psi \Rightarrow \varphi}\ {\scriptstyle \mathrm{Id}}}{\Box\varphi \Rightarrow \psi, \psi \to \varphi}\ {\scriptstyle \to R'}$$

This alternative rule is compatible with the measure decreasing in G4iSLt, entailing the termination of naive backward proof search in our calculus, which we call G4KM. To use Pitts' technique, we are left to show that cut is admissible in G4KM, a result we are currently formalising in Rocq. To get there, we intend to use a well-established direct argument for G4ip [4] which has already been adapted to the multi-succedent setting [2].

Computing uniform interpolants. Assuming the admissibility of cut, and given the strong termination of G4KM, we seemingly found the ideal candidate to apply Pitt's technique and compute uniform interpolants for KM. However, we expect the road to the application of this technique to G4KM to be a bumpy one. Indeed, the computation of interpolants follows the structure of the proof-search tree of a sequent, which is made larger by multi-succedent sequents of G4KM. The cases for computation should thus grow in number, and the correctness argument showing that the computed formula is an interpolant will follow this trend. Still, the porting of this technique to a multi-succedent setting is of intrinsic interest, as it would give a framework to treat logics calling for such calculi, e.g. intuitionistic modal logics with a normal diamond.

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