▸ LUIZ CARLOS PEREIRA, VICTOR BARROSO-NASCIMENTO, AND ELAINE PIMENTEL, *Three alternative systems with alternatives.*
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In 2015 Dag Prawitz proposed an Ecumenical system (we call it $\mathcal{NE}$) where classical and intuitionistic logic could coexist in peace (see [Pra15]). The classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation and the constant for the absurd, but they would each have their own existential quantifier, disjunction and implication, with different meanings. Prawitz's main idea is that these different meanings are given by a semantical framework that can be accepted by both parties. Prawitz's rules for the classical operators $\vee_c$ and $\to_c$ have been criticized for not being *pure* in the sense that negation is used in the definition of the introduction and elimination rules for the classical operators:

1. The rules for $\vee_c$ are:

$$\frac{\begin{array}{c}[\neg A] \quad [\neg B] \\ \Pi \\ \bot\end{array}}{(A \vee_c B)}\ \vee_c\text{-Int} \qquad \frac{(A \vee_c B) \quad \begin{array}{c}[A]\\ \Pi_1 \\ \bot\end{array} \quad \begin{array}{c}[B]\\ \Pi_2 \\ \bot\end{array}}{\bot}\ \vee_c\text{-Elim}$$

2. The rules for $\to_c$ are:

$$\frac{\begin{array}{c}[A] \quad [\neg B] \\ \Pi \\ \bot\end{array}}{(A \to_c B)}\ \to_c\text{-Int} \qquad \frac{(A \to_c B) \quad A \quad \begin{array}{c}[B]\\ \Pi_2 \\ \bot\end{array}}{\bot}\ \to_c\text{-Elim}$$

and we can clearly see that these rules use negated formulas $(\neg A, \neg B)$ in order to introduce/eliminate $\vee_c$ and $\to_c$.

There are many ways of proposing pure, harmonic natural deduction systems for (propositional) classical logic. Indeed, Murzi [6] proposes a new set of rules for classical logical operators based on absurdity as a punctuation mark, and higher-level rules [12]. D'Agostino [1] brings a totally different approach, presenting a theory of classical natural deduction that makes a distinction between operational rules, governing the use of logical operators, and structural rules dealing with the metaphysical assumptions governing the (classical) notions of truth and falsity, namely the principle of bivalence and the principle of non-contradiction.

Yet another approach appears in [2], where Michael and Murdoch Gabbay present the natural deduction version of Dov Gabbay's *Restart* rule

$$\frac{A}{B}\ \textit{Restart}$$

with the side-condition that, below every occurrence of *Restart* from A to B, there is (at least) one occurrence of A. The intended meaning is that B is a new start to a line of reasoning concluding A. For example, in the derivation of the Peirce's Law

$$\dfrac{\dfrac{[(A \to B) \to A)] \qquad \dfrac{\dfrac{[A]}{B}\,\textit{Restart}^\star}{A \to B}\to\text{-int}}{A^\dagger}\to\text{-elim}}{((A \to B) \to A) \to A)}\to\text{-int}$$

the restart at * is justified at $\dagger$.

In [8] we adapted Parigot and Girard's *stoup* mechanism to the ecumenical setting. This allowed the definition of a pure harmonic natural deduction system for the propositional fragment of Prawitz' ecumenical logic. In that system, the nodes in derivations have the form $\Delta; \Sigma$, where Δ is a set or multiset of formulas, called the *classical context*, and Σ, the *stoup*, is a multiset containing at most one formula. In this formulation the premise formula of the stoup moves in/out the classical context via the *dereliction* and *store* rules:

$$\dfrac{\Delta; A}{\Delta, A; B}\text{ der} \qquad\qquad \dfrac{\Delta, A; \cdot}{\Delta; A}\text{ store}$$

Considering Δ a set, the proof of Peirce's Law using *stoup* has the following general form

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \to B) \to A)] \qquad \dfrac{\dfrac{[A]}{A; B}\text{ der}}{A; A \to B}\to\text{-int}}{A; A}\to\text{-elim}}{A; \cdot}\text{ der}}{\cdot; A}\text{ store}}{\cdot; ((A \to B) \to A) \to A)}\to\text{-int}$$

On the other hand, natural deduction systems with alternatives were introduced by Greg Restall in [11]. These new formalisms extend Gentzen-Prawitz-style natural deduction systems by means of the addition of a single structural mechanism: derivations can now use negatively signed assumptions, called *alternatives*. In a nutshell, the technique of alternatives uses the *store* rule:

$$\dfrac{A \qquad \cancel{A}}{\bot}\uparrow$$

with the following interpretation: having a proof Π of A, one can *set A aside*. In the the remaining of the proof, the conclusion A is not ignored, but added to the collection of alternatives current at this point of the proof, which can then be used at some point of the proof. Restall used this framework to obtain (1) classical logic, (2) relevant logic without distribution, (3) affine logic, and (4) linear logic. We can use this approach in order to re-visit Prawitz's system. In fact, the rules $\vee_c$-elimination and $\to_c$-elimination are the same as in Prawitz' system.

The rules for $\vee_c$ and $\to_c$ in the system $E - alt$ are:

$\vee_c$-Introduction

$$\dfrac{\begin{array}{cc}[\cancel{A}]^n & [\cancel{B}]^m \\ \multicolumn{2}{c}{\Pi} \\ \multicolumn{2}{c}{\bot}\end{array}}{(A \vee_c B)}\; n, m$$

$\to_c$-Introduction

$$\dfrac{\begin{array}{cc}[A]^n & [\cancel{B}]^m \\ \multicolumn{2}{c}{\Pi} \\ \multicolumn{2}{c}{\bot}\end{array}}{(A \to_c B)}\; n, m$$

The aim of this (ongoing) work is to define a new pure ecumenical system $E - alt$ with the use of Greg Restall's notion of *alternatives*. We shall also show how to use Restall's proposal to obtain:

1. a new single conclusion natural deduction system for the intermediate logic of constant domains CD [4]
2. a natural deduction codification of Jean-Baptiste Joinet's disymmetrical linear logic DLL [5].

[1] Marcello D'Agostino, *Classical Natural Deduction*, ***We Will Show Them! Essays in Honour of Dov Gabbay, Volume One***, In Sergei N. Artëmov, Howard Barringer, Artur S. d'Avila Garcez, Luís C. Lamb and John Woods, College Publications (2005), pp. 429–468.

[2] Michael Gabbay and Murdoch Gabbay, *Some formal considerations on Gabbay's restart rule in natural deduction and goal-directed reasoning*, ***We Will Show Them! Essays in Honour of Dov Gabbay, Volume One***, In Sergei N. Artëmov, Howard Barringer, Artur S. d'Avila Garcez, Luís C. Lamb and John Woods, College Publications (2005), pp. 701–730.

[3] Jean-Yves Girard, *A new constructive logic: Classical logic*, ***Mathematical Structures of Computational Science***, vol 1:3 (1991), pp. 255–296.

[4] Andrzej Grzegorczyk, *A Philosophically Plausible Formal Interpretation of Intuitionistic Logic*, ***Indagationes Mathematicae***, vol. 26 (1964), pp. 596–601.

[5] Jean-Baptiste Joinet, *Dissymetrical Linear Logic*, `https://univ-lyon3.hal.science/hal-03815339`

[6] Julien Murzi, *Classical harmony and separability*, ***Erkenntnis***, vol. 85 (2020), pp. 391–415.

[7] Luiz Carlos Pereira and Ricardo Oscar Rodriguez, *Normalization, soundness and completeness for the propositional, fragment of Prawitz' ecumenical system*, ***Revista Portuguesa de Filosofia***, vol. 73:3-3 (2017), pp. 1153–1168.

[8] Luiz Carlos Pereira and Elaine Pimentel, *On an ecumenical natural deduction with stoup. part I: the propositional case*, In Antonio Piccolomini d'Aragona, editor, ***Perspectives on Deduction: Contemporary Studies in the Philosophy, History and Formal Theories of Deduction***, vol. 481 (2024), Synthese Library, Springer, pp. 139–169.

[9] Dag Prawitz, *Natural Deduction*, ***Stockholm Studies in Philosophy***, vol. 3 (1965), Almqvist and Wiksell.

[10] Dag Prawitz, *Classical versus intuitionistic logic*, In Bruno Lopes Edward Hermann Haeusler, Wagner de Campos Sanz, editor, ***Why is this a Proof?, Festschrift for Luiz Carlos Pereira***, vol. 27 (2015), College Publications, pp.15–32.

[11] Greg Restall, *Structural rules in Natural Deduction with alternatives*, ***Bulletin of the Section of Logic***, vol. 52/2 (2023), pp. 109–143. `https://doi.org/10.18778/0138-0680.2023.06`

[12] Peter Schroeder-Heister, *The calculus of higher-level rules, propositional quantification, and the foundational approach to proof-theoretic harmony*, ***Studia Logica***, vol. 102:6 (2014), pp. 1185–1216.