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The Modal Cube Revisited: Semantics without Worlds.
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Modal logics are built on top of propositional classical logic (CL) by introducing the modal operators $\Box$ and $\Diamond$, which, under the alethic interpretation, correspond to *necessary* and *possible*, respectively. Thus from an atomic sentence p, which in CL has value either false (**F**) or true (**T**), one can construct the statements *necessarily* p and *possibly* p, denoted by $\Box p$ and $\Diamond p$, respectively.

When interpreting these formulas in Kripke semantics, modalities *qualify* the notion of truth, which now depends on the world in which p is being evaluated. Kripke's concept of *possible worlds* thus provides an elegant framework for capturing the possible values of propositions across different worlds, each with its own *truth table*. This natural generalization of the straightforward semantics of CL makes it an ideal foundation for the semantics of modal logics.

A natural question that arises is whether one could combine these tables to obtain a single (finite-valued, maybe non-deterministic) truth table capable of describing modalities. The answer is negative, as noted by Dugundji in [6], which largely halted the study of truth tables for modal logics.

In an effort to circumvent Dugundji's result–and as a compelling alternative to Kripke semantics–Kearns [9] and Ivlev [8] independently developed frameworks for characterizing certain modal systems[1]. Kearns used four-valued multivalued truth-functions[2], and introduced a constraint on valuations known as *level-valuations* to characterize the modal systems KT, S4, and S5. The idea is that the level-0 corresponds to the standard distribution of truth values (just following the corresponding matrix), while higher levels filter out valuations that violate the necessitation rule, preserving only those that assign the designated truth value to tautologies. Validity is therefore defined as the (infinite) intersection of all level valuations, corresponding to the closure w.r.t. necessitation. This layered structure gives to necessity a *global interpretation*.

Kearns' (and Ivlev's) contributions remained largely overlooked and relatively obscure until they were independently revisited in two separate lines of research. In [3], Coniglio, Fariñas del Cerro, and Peron reconstructed and extended these earlier results, proposing a characterization of the systems KB using four-valued Nmatrices with level-valuations. They also introduced six-valued Nmatrices for a range of modal systems, including KT, S4, S5, KD, KDB, KD4, and KD45. Around the same time, Omori and Skurt, in [14], also began by revisiting Kearns-Ivlev's original framework, and proposed an extension using eight truth values capable of capturing the modal logic K and

[1] Ivlev proposed semantics for a range of non-normal modal systems lacking the necessitation rule–representing weaker versions of KT and S5 (later extended in, *e.g.*, [12, 13]). In this work, we will focus on the extension of Kearn's work to the (normal) modal cube. A related approach can be found in [5].

[2] The notion of multivalued truth-functions was later formalized by Avron and Lev under the term *Nmatrices* [1]. See also [2].

six truth values for KTB. While there is considerable conceptual overlap between [3] and [14], the two works were developed independently.

Although conceptually interesting, these works have limited practical applicability, as determining level-valuations requires accounting for the valuations of *all* tautologies across *all* levels. This requires a twofold infinite testing: on formulas *and* levels.

This situation remained unchanged until the work of Grätz [7], whose key contribution was the introduction of a *decision procedure* for Nmatrices that is both sound and complete w.r.t. Kearns' semantics. This breakthrough brought renewed attention to an otherwise underexplored area, paving the way for extending the methods of Kearns, Ivlev, and Grätz to a broader class of logics. For instance, [10] presents a semantic characterization of propositional intuitionistic logic (IPL) using a three-valued non-deterministic matrix with a restricted set of valuations, enabling a remarkably simple decision procedure for IPL.

This work follows this path by providing a decision procedure based on multi-valued non-deterministic matrices for *all logics* in the modal cube. In particular, our approach focuses on the following key aspects.

Modularity. The results in [3, 14, 4] share a common limitation: a lack of modularity. These works aim to "explain" or "refine" Kearns' original approach, not addressing the fundamental challenge of uniformly extending it. In fact, different systems use different sets of truth-values, often diverging significantly from standard choices. An alternative axiomatization was proposed by Pawlowski and la Rosa [11], resulting in a modular rule of necessitation to KT, KTB, S4, and S5 (and some non-normal modal logics).

In contrast, our work adopts a fundamentally different approach: we begin with a uniform set of eight truth values and systematically develop level-valuations for *all* logics in the modal cube. We demonstrate that, under certain axioms, some of these truth values are eliminated, thereby recovering many of the semantics proposed in the *op. cit.* works. Our definitions are guided by a *modal characterization* of the truth values, which enables modular procedures for proving soundness and completeness of the semantics, thus unifying and generalizing existing systems in the literature. Notably, this generalization allowed for new level-semantics for K4, K5, K45, KD4, KD5 and KB5.

Decision procedure. We propose new Nmatrix-based decision procedures for the entire modal cube. This is achieved uniformly via the modal characterization of truth values. We thus extend Grätz's work on KT and S4 to all 15 normal modal logics, thereby completing the picture for the entire modal cube. This uncovers the undeniable potential of this alternative semantics.

Modal Semantics without Possible Worlds? Kearns concludes his paper with the following striking statement [9, p. 86]:

> "The present semantic account [...] is simpler than the standard account in virtue of having dispensed with possible worlds and their relations. I also think that my account is philosophically preferable to the standard account for having done this. For I do not think there are such things as possible worlds, or even that they constitute a useful fiction."

Unfortunately, the price of rejecting possible worlds may seem steep: one must contend with multi-valued truth values, non-deterministic matrices, and–prior to the development of decision procedures–at least two levels of valuations. As a final contribution, we reestablish the connection with Kripke semantics by linking matrix filters to Kripke models, thus providing a more "ecumenical" perspective where the two semantics can coexist. We thus settle a longstanding conjecture posed by Omori and Skurt [14, p. 27]:

> "One of the virtues of Kripkean semantics is the correspondence between axioms and the accessibility relations of the Kripke frame [...] But a glance at the [Kearnsean] semantics for the systems [...] introduced here reveals that *if there is a correspondence it is not a simple one*."

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