Interpolation for the two-way modal μ-calculus

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The *modal μ-calculus* $\mathcal{L}_\mu$ extends modal logic with least and greatest fixpoint operators, enabling the expression of various recursive concepts in the language. The *two-way modal μ-calculus* $\mathcal{L}^2_\mu$ is a natural extension of the modal μ-calculus obtained by adding, for each modality a, a modality $\breve{a}$, which in the semantics will be interpreted as the converse of the accessibility relation for a. This addition enables the logic to reason about the past, which is for instance attractive from the perspective of formal program verification.

Compared to its one-way version, surprisingly little seems to be known about the two-way modal μ-calculus. While it is not hard to see that the logic lacks the finite model property, a key result by Vardi [6] states that the satisfiablity problem for $\mathcal{L}^2_\mu$ can be solved in exponential time. Vardi introduces the notion of an alternating two-way tree automaton, and the key argument in his proof is based on a reduction of these two-way automata to one-way deterministic tree automata.

Our goal is to contribute to the knowledge of the two-way modal μ-calculus by showing that it has the (local) *Craig interpolation property*: For any pair of formulas φ and ψ such that ψ is a consequence of φ (denoted as $\varphi \vDash \psi$) one may find an *interpolant*; that is, a formula θ in the common vocabulary of φ and ψ such that $\varphi \vDash \theta$ and $\theta \vDash \psi$. As a corollary we also obtain that the logic enjoys *Beth definability*.

We show this result *proof-theoretically*. In order to prove the interpolation property we introduce two sound and complete proof systems for $\mathcal{L}^2_\mu$. The first proof system NW^2 permits proofs with infinite branches and builds on the non-wellfounded tableau system for $\mathcal{L}_\mu$ of Niwiński & Walukiewicz [3]. The extension with backward modalities causes two kinds of challenges. The first complication is that cut-free derivation systems for modal logics with backward modalities have to go beyond simple sequent systems, even in the absence of fixpoint operators. Our solution to this problem is to allow applications of cut that are *analytic*, in the wide sense that the cut formula must be taken from some bounded set of formulas.

The second and main challenge is to formulate adequate *success conditions* on infinite proof branches. The problem is that the combinatorics of the formula traces are more complicated than in the one-way setting, since traces may move both up and down a proof tree. In order to deal with this issue we follow Rooduijn & Venema [4], who enrich the syntax of their proof calculus with so-called *trace atoms*. Roughly speaking, trace atoms hardwire the ideas underlying Vardi's two-way automata explicitly into the syntax.

The second proof system JS^2 is *cyclic* and features annotations in the style of Jungteerapanich [1] and Stirling [5]. For showing soundness and completeness of JS^2 we employ an ω-automaton that checks the success-condition on infinite branches in an NW^2 proof.

We establish *interpolation* for $\mathcal{L}^2_\mu$ by adapting Maehara's method to the cyclic system JS^2. In this approach one takes some finite proof of an implication and defines interpolants for each node of the derivation tree by means of a leaf-to-root induction. The challenge in a cyclic system is that some proof leaves are not axiomatic and hence fail to have a trivial interpolant. However, each such leaf is discharged at a companion node closer to the root. The idea is now to associate a fixpoint variable with each discharged leaf and to bind this variable at the companion with a fixpoint operator.

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