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Justification logics [1, 2, 4] which refine modal logics by replacing modal operators with explicit *proof terms*. In a similar way to how a modal formula $\Box A$ can be read as A *is provable* in provability logics, it can be given an explicit justification counterpart $t{:}A$ for some proof term t, to be read as *there exists a proof t of A*. A *realisation theorem* provides a formal connection between a justification logic and its corresponding modal logic: each theorem of the modal logic is translated into a corresponding theorem of the justification logic by *realising* every modality with proof terms. This can be achieved proof-theoretically through a close inspection of proofs in a cut-free proof system.

Soma intuitionistic variants of justification logic have been suggested [3, 5, 9, 20, 16, 17, 14]. However none of these works provide counterparts to an intuitionistic modal logic which contains both the box and the diamond modalities.

The first justification logic for an intuitionistic modal logic which makes the diamond explicit was postulated in [15] as a justification counterpart to *constructive modal logic* CK [19, 7, 6] via a syntactic realisation procedure. As $\Box$ is replaced by a proof term t; so is the diamond operator $\Diamond$ replaced with a *satisfier* term μ. We expand this line of work to provide a justification counterpart to Fischer Servi's intuitionistic modal logic IK, as defined originally by [10, 11, 12, 18] (see axioms in Figure 1).

Formally *proof terms* $t, s, \ldots$ and *satisfiers* $\mu, \nu, \ldots$ are generated as follows:

$$t ::= x \mid c \mid (t + t) \mid (t \cdot t) \mid (\mu \triangleright t) \mid !t \qquad \mu ::= \alpha \mid (\mu \sqcup \mu) \mid (t \star \mu)$$

with x ranges over proof variables, α over satisfier variables, and c over proof constants. The operations *proof sum* $+$, *application* $\cdot$ and *proof checker* ! are the usual justification operations relating to proof manipulations. The operations of *propagation* $\star$ and *disjoint union* $\sqcup$ were introduced in [15] to consolidate the intuition that in $\mu{:}A$, μ is some *model* of A. The operation $\star$ can be seen as a combination of local and global reasoning, e.g. reading $\mu{:}A$ as A is a local fact, we can use the proof t of $A \to (A \vee B)$ to locally reason $t \star \mu{:}(A \vee B)$. The operation $\sqcup$ is akin to a disjoint union of models. We introduce here the operation *local update* $\triangleright$ which carries the intuition that if a local fact implies global knowledge, one can update the global knowledge with local data.

We define the *justification counterpart of modal logic* IK, which we call JIK, as the extension of IPL with axioms $\mathsf{jk}_1 - \mathsf{jk}_5$, $\mathsf{j}+$ and $\mathsf{j}\sqcup$ from Figure 1, together with the *constant axiom necessitation rule*: $\mathsf{can}\,\dfrac{}{c_n{:}\ldots{:}c_1{:}A}$ where $c_1, \ldots, c_n$ are proof constants and A is any justification axiom instance in Figure 1.

We furthermore define justification counterparts to the extensions of IK introduced

$$\begin{array}{lll}
\mathsf{k}_1 : \Box(A \to B) \to (\Box A \to \Box B) & & \\
\mathsf{k}_2 : \Box(A \to B) \to (\Diamond A \to \Diamond B) & \mathsf{t}_\Box : \Box A \to A & \\
\mathsf{k}_3 : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B) & \mathsf{t}_\Diamond : A \to \Diamond A & \mathsf{nec}\,\dfrac{\vdash A}{\vdash \Box A} \\
\mathsf{k}_4 : (\Diamond A \to \Box B) \to \Box(A \to B) & \mathsf{4}_\Box : \Box A \to \Box\Box A & \\
\mathsf{k}_5 : \Diamond\bot \to \bot & \mathsf{4}_\Diamond : \Diamond\Diamond A \to \Diamond A &
\end{array}$$

$$\begin{array}{lll}
\mathsf{jk}_1 : s{:}(A \to B) \to (t{:}A \to s \cdot t{:}B) & & \\
\mathsf{jk}_2 : s{:}(A \to B) \to (\mu{:}A \to s \star \mu{:}B) & \mathsf{j}+ : s{:}A \to (s+t){:}A & \mathsf{jt}_\Box : t{:}A \to A \\
\mathsf{jk}_3 : \mu{:}(A \vee B) \to (\mu{:}A \vee \mu{:}B) & \mathsf{j}+ : t{:}A \to (s+t){:}A & \mathsf{jt}_\Diamond : A \to \mu{:}A \\
\mathsf{jk}_4 : (\mu{:}A \to t{:}B) \to \mu \triangleright t{:}(A \to B) & \mathsf{j}\sqcup : \mu{:}A \to (\mu \sqcup \nu){:}A & \mathsf{j4}_\Box : t{:}A \to !t{:}t{:}A \\
\mathsf{jk}_5 : \mu{:}\bot \to \bot & \mathsf{j}\sqcup : \nu{:}A \to (\mu \sqcup \nu){:}A & \mathsf{j4}_\Diamond : \mu{:}\nu{:}A \to \nu{:}A
\end{array}$$

FIGURE 1. Intuitionistic modal and justification axioms

$$
\bot^\bullet\ \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{id}\ \frac{}{\Lambda\{p^\bullet, p^\circ\}} \qquad \mathsf{c}^\bullet\ \frac{\Gamma\{\Lambda, \Lambda\}}{\Gamma\{\Lambda\}} \qquad \wedge^\bullet\ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}
$$

$$
\wedge^\circ\ \frac{\Lambda\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}} \qquad \vee^\bullet\ \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee_1^\circ\ \frac{\Lambda\{A^\circ\}}{\Lambda\{A \vee B^\circ\}} \qquad \vee_2^\circ\ \frac{\Lambda\{B^\circ\}}{\Lambda\{A \vee B^\circ\}}
$$

$$
\to^\bullet\ \frac{\Gamma^\downarrow\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to B^\bullet\}} \qquad \to^\circ\ \frac{\Lambda\{A^\bullet, B^\circ\}}{\Lambda\{A \to B^\circ\}} \qquad \Box^\bullet_{[\cdot]}\ \frac{\Lambda\{[A^\bullet, \Gamma]\}}{\Lambda\{\Box A^\bullet, [\Gamma]\}}
$$

$$
\Box^\bullet_{\langle\cdot\rangle}\ \frac{\Gamma\{\langle A^\bullet, \Lambda\rangle\}}{\Gamma\{\Box A^\bullet, \langle\Lambda\rangle\}} \qquad \Box^\circ\ \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}} \qquad \Diamond^\bullet\ \frac{\Gamma\{\langle A^\bullet\rangle\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Diamond^\circ\ \frac{\Lambda_1\{[A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond A^\circ, \langle\Lambda_2\rangle\}}
$$

$$
\mathsf{t}^\bullet\ \frac{\Gamma\{A^\bullet\}}{\Gamma\{\Box A^\bullet\}} \qquad \mathsf{t}^\circ\ \frac{\Lambda\{A^\circ\}}{\Lambda\{\Diamond A^\circ\}}
$$

$$
\mathsf{4}^\bullet_{[\cdot]}\ \frac{\Lambda\{[\Box A^\bullet, \Gamma]\}}{\Lambda\{\Box A^\bullet, [\Gamma]\}} \qquad \mathsf{4}^\bullet_{\langle\cdot\rangle}\ \frac{\Gamma\{\langle\Box A^\bullet, \Lambda\rangle\}}{\Gamma\{\Box A^\bullet, \langle\Lambda\rangle\}} \qquad \mathsf{4}^\circ\ \frac{\Lambda_1\{[\Diamond A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond A^\circ, \langle\Lambda_2\rangle\}}
$$

FIGURE 2. System nIK and modal rules for t and 4

previously using the additional axioms on Figure 1: $\mathsf{JIKt} = \mathsf{JIK} + \mathsf{jt}_\Box + \mathsf{jt}_\Diamond$, $\mathsf{JIK4} = \mathsf{JIK} + \mathsf{j4}_\Box + \mathsf{j4}_\Diamond$ and $\mathsf{JIS4} = \mathsf{JIK} + \mathsf{jt}_\Box + \mathsf{jt}_\Diamond + \mathsf{j4}_\Box + \mathsf{j4}_\Diamond$. We will write L for any logic in $\{\mathsf{IK}, \mathsf{IKt}, \mathsf{IK4}, \mathsf{IS4}\}$ and JL for the corresponding logic. We will write $\mathcal{L}_\Box$ for the language of modal logics and $\mathcal{L}_\mathsf{J}$ for the language of justification logics.

The connection from JL to L is directly achieved through the forgetful projection $(\cdot)^\mathsf{f} : \mathcal{L}_\mathsf{J} \to \mathcal{L}_\Box$ inductively defined as follows, where $* \in \{\wedge, \vee, \to\}$:

$$
\bot^\mathsf{f} := \bot \qquad p^\mathsf{f} := p \qquad (A * B)^\mathsf{f} := (A^\mathsf{f} * B^\mathsf{f}) \qquad (t{:}A)^\mathsf{f} := \Box A^\mathsf{f} \qquad (\mu{:}A)^\mathsf{f} := \Diamond A^\mathsf{f}
$$

THEOREM 1. *Let* $A \in \mathcal{L}_\mathsf{J}$. *If* $\mathsf{JL} \vdash A$ *then* $\mathsf{L} \vdash A^\mathsf{f}$.

Following from the fact that the forgetful projection on axioms of JL are theorems of L.

The more interesting direction is establishing a connection from L to JL. This is done through using a *realisation* function $(\cdot)^r : \mathcal{L}_\Box \to \mathcal{L}_\mathsf{J}$ such that $(A^r)^\mathsf{f} = A$ for each $A \in \mathcal{L}_\Box$. The condition $(A^r)^\mathsf{f} = A$ ensures that each $\Box$ and $\Diamond$ term is replaced with exactly one proof or satisfier term. The connection is then stated formally as follows:

THEOREM 2 (Realisation Theorem). *Let* $A \in \mathcal{L}_\Box$. *If* $\mathsf{L} \vdash A$, *then there exists a realisation* r *such that* $\mathsf{JL} \vdash A^r$.

We adapt the methods used in [8, 13]. Starting from a derivation of A in the cut-free intutionistic nested sequent system nL of [21] (see Figure 2) for L, the idea is to construct a realisation r on A by induction on the height of the derivation in nL.

1. For the base case, the derivation is an instance of $\bot^\bullet$ or id. Each step of the Hilbert proof of their soundness can be transformed into its justification counterpart, and doing so, replaces $\Box$s and $\Diamond$s with proof terms and satisfiers.
2. For the inductive case, we look at the last rule used in the derivation
$$
\mathsf{rule}\ \frac{\Gamma_1 \quad \cdots \quad \Gamma_n}{\Gamma}
$$
Using the inductive hypothesis, we have some realisations $r_1, \ldots r_n$ with $\mathsf{JL} \vdash {\Gamma_1}^{r_1}, \ldots, \mathsf{JL} \vdash {\Gamma_n}^{r_n}$. Using these theorems, we construct a realisation r on Γ, similarly utilising the Hilbert proof of the soundness of rule. Here, there is added difficulty as the soundness of nested rules make use of both modal and propositional reasoning which are normally separated in normal Gentzen-style rules.

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